Gain/loss Asymmetry and the Leverage Effect

MSc MTEC Master Thesis
Yannick LAGGER
August 2011 - January 2012

Tutor: Prof. Dr. D. Sornette
Supervisor: Dr. V. Filimonov
Department of Management, Technology & Economics - ETH Zürich
Abstract

In this paper we establish empirical relations between gain/loss asymmetry and the leverage effect, both at the individual stock and at the index level. Both are stronger in indices than in individual stocks and they both display a rebound effect: a positive correlation between past volatility and present returns. The leverage effect decays faster in indices than in individual stocks and the maximum occurs one day earlier, at $\tau = 0$ for the individual stocks. By building an artificial equity index, we show that both of these stylized facts are amplified compared to their average value in the constituents. This leads to the central hypothesis of this work: ‘asset cross-correlation is sufficient for the amplification of gain/loss asymmetry and the leverage effect in indices compared to individual stocks’. With a model based on EGARCH processes, we show how gain/loss asymmetry and the leverage effect are increased in an index in function of the asset cross-correlation and underline the dynamics of the amplification. Finally, using the same model specification, we show what mechanism is responsible for the offset in the leverage effect for stocks and how a slower decay of the leverage effect function leads to stronger gain/loss asymmetry.
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Chapter 1

Introduction

Both physicists and economists have been studying and analyzing complex time series with forward statistics for many years. Among the many problems encountered in physics, the motion of fluids and more precisely the phenomenon of turbulence represents one of the oldest unsolved problems. Turbulence describes an apparent random motion of fluid molecules moving among each other, in which some short and violent ‘bursts’ appear. The traditional way of describing turbulence is based on the initial theory developed by Kolmogorov in 1941 [1]. In this theory, which is unfortunately incomplete, the relative velocity difference between two neighboring elements, separated by a given distance, is estimated and averaged over time and space. This leads to co-called structure functions, which describe how the averaged velocity scales with the separation distance. These functions have been the central element of turbulence research over the past five decades [59]. This kind of analysis was termed forward statistics.

Forward statistics have been thoroughly applied to financial time series as well. Actually, the spikiness behavior of fluid molecules in turbulence, as described above, is also observed in financial time series. They also display a turbulent like motion, with regular bursts of strong activity, sometimes termed intermittency. In finance, the forward question can be formulated as: ‘Choosing a given time scale, what is the typical, average return over that period?’ With a varied time scale, non-normal distributions of returns have been obtained, which display ‘fat-tails’ of high probability [19].

In order to obtain a deeper understanding of the fluctuations in turbulence and to supplement the established literature with alternative measures, Jensen suggested the use of inverse statistics. A different measure of a known phenomenon provides novel and useful diagnostics, which may confirm or falsify existing models and enhance present understanding. In the field of turbulence, Jensen suggested approaching the problem from
another point of view and asked an inverse question: ‘For a given velocity difference between two fluid molecules, what is the typical, averaged, distance where such a velocity difference is obtained for the first time?’ [28]. Performing this analysis leads to so-called inverse structure functions, which provide alternate information about the temporal behavior. Whereas forwards structure functions measure the spikes, the inverse structure functions measure the quiet, laminar states.

The idea of inverse statistics has been applied in the analysis of financial data. For financial time series the inverse question can be written as follows: ‘For a given return on an investment, what is the typical, average time span needed to obtain this return (either positive or negative)?’ This question was investigated in a series of papers [38, 41, 42, 45] and is closely related to the first passage time problem. The latter describes when a barrier is crossed for the first time, starting from a fixed value. The key difference is that inverse statistics consider many different starting levels, over which an average is calculated. Similarly to turbulence, when inverse statistics are applied to financial time series, one finds distributions with a sharp maximum and a long ‘fat-tail’ [38]. For financial indices however, an additional feature was discovered: while the maximum of the inverse statistics for a positive level of return occurs at a specific time, the maximum of the inverse statistics for the same negative level of return appears earlier. The difference was found to be larger in indices than in individual stocks [54], if the return level chosen is scaled to the volatility of the data. This is the origin of the gain/loss asymmetry of financial indices, which is a central topic in this paper.

The leverage effect is another stylized fact of financial time series and may be related to gain/loss asymmetry. There are several established ‘stylized facts’ of financial time series, such as heavy tails, or long-ranged volatility autocorrelation (see Cont 2001 for a review [34]). Among those, the leverage effect was first discussed by Black [4, 6], who observed that volatility tends to increase after a price drop. Although widely discussed in the literature [29, 57], the leverage effect hasn’t been systematically and quantitatively investigated before Bouchaud et al. in 2001 [33]. The paper shows that correlations are between past price changes and future volatilities. Furthermore, they show that the leverage effect has a stronger amplitude and faster decay in indices than in stocks.

The aim of this work is to study the relation between the leverage effect and gain/loss asymmetry, and what mechanisms lead to its amplification in equity indices. Ahlgren et al. were the first to underline the common features between gain/loss asymmetry and the leverage effect [47]. Furthermore, Siven and Lins show with an EGARCH model that the gain/loss asymmetry observed is linearly related to a model parameter responsible for the leverage effect [54]. In this paper we try to establish empirical relations
between these two stylized facts, both at the individual stock and at the index level. With a model based on EGARCH processes, we study how gain/loss asymmetry and the leverage effect is altered when measured as part of an index or individually. Finally, using the same model specification, we investigate other mechanisms which may reproduce these stylized facts.

The master’s thesis is organized in four parts. First, we review the actual state of the art and introduced the ARCH family of models, which is used in the following chapters. In the main chapter ‘Gain/Loss asymmetry and the Leverage Effect’ we contribute to the existing literature in three ways: the links between the two stylized facts are reviewed empirically (both at the index and individual level), a correlation model details the mechanisms leading to an amplification of these properties in an index, and finally we investigate two open questions relating to the leverage effect. In the end a conclusion is drawn and some future perspectives describe what kind of work could be undertaken after this thesis.
Chapter 2

Theoretical Background

Forward statistics investigate the level of return that can be obtained over a fixed time frame, whereas evaluating the typical average time span required to reach a given level of return is called inverse statistics. This ‘inverse’ approach was first applied in the field of turbulence [28] before being applied to financial data [38, 40, 42, 45]. Although this approach is very similar to the first passage time problem, the subtle difference is that any given starting point is considered in inverse statistics. A distribution of ‘waiting times’ is calculated by averaging over these inverse statistics from many different starting levels and yields.

Many years of empirical data analysis show that financial time series display remarkably common properties called stylized facts. In an attempt to explain or rationalize a given market movement, market analysts often adopt a event-based approach [11]. This method consists of linking the observed price change to an economical or political event or announcement and trying to extract a causal relationship between them. Because different assets, industries, or markets are not necessarily influenced by the same events or informations, one can expect that they will exhibit different properties. Nonetheless, numerous empirical findings show that these seemingly random variations of asset prices share some interesting statistical properties [34]. This collection of properties, similar across a wide range of methods, markets and time periods are called ‘stylized empirical facts’.

Financial models are abstract constructs representing asset prices or returns by a set of variables and relationships between them. For example, synthetic time series can be created to study the effect of the covariance structure of asset returns on different stylized facts. Among the many different model families and variations, we will focus on the specification defined by Nelson [12] of the Exponential Generalized Autoregressive Conditional Heteroskedastic (EGARCH) model, which offers a solid tool to explore the stylized facts of interest.
2. Theoretical Background

The first section of this chapter presents in detail the idea of inverse statistics. Some stylized facts of asset returns, with a greater emphasis on gain/loss asymmetry and the leverage effect, are presented in the second section. Finally, the Autoregressive Conditional Heteroskedastic (ARCH) model family and one of its specifications used in this work are decreed in the third section.

2.1 Inverse Statistics

Performance and risk of holding a certain asset is traditionally measured by studying the probability distribution of returns extracted from historical data over a fixed time span. In finance time series are directly observable from stock indices, individual stock quotes, interest rates, exchange rates, etc. Based on this data, usually one fixes a time scale and seeks the typical, average return that is obtained over that period. For example, the non-normal behavior of the distribution of returns and the ‘fat tail’ property have been discovered in this straightforward manner [2,19].

Using \( S(t) \) to denote price of a stock at time \( t \), the logarithmic return at time \( t \) over a time window \( \Delta t \) can be defined as [30,32]:

\[
    r_{\Delta t}(t) = s(t + \Delta t) - s(t)
\]

where \( s(t) = \ln S(t) \) is the logarithmic asset price. Therefore, the forward statistics approach studies the variations of \( r_{\Delta t}(t) \) over time, given a fixed time window \( \Delta t \).

Because it is not always possible to integrate ‘fat tailed’ distributions, one could attack the problem by the other end and evaluate the typical average time span required to reach a given level of return, an approach referred to as inverse statistics [38]. Contrary to forward statistics, the level of return \( \rho \) is kept fixed and one looks for the shortest waiting time after \( t \), \( \tau_{\pm \rho}(t) \), for which the logarithmic return is above or below a predefined level \( \pm \rho \).

Using this notation, \( \rho \) is the return level, the positive or negative sign corresponds to a gain, or a loss respectively. Figure 2.1 shows schematically the inverse statistics method. Mathematically, the first passage times are defined as:

\[
    \tau_{\pm \rho}(t) = \begin{cases} 
        \inf \{ \Delta t \mid r_{\Delta t} \geq +\rho \}, \\
        \inf \{ \Delta t \mid r_{\Delta t} \leq -\rho \}.
    \end{cases}
\]  

(2.2)

In the inverse statistics framework, the distributions of the waiting times for gains \( \tau_{+\rho} \) and losses \( \tau_{-\rho} \), is denoted \( p(\tau_{\pm \rho}) \) and corresponds to a fixed return level \( \rho \).
2.1. Inverse Statistics

Figure 2.1: Schematic overview of inverse statistics on logarithmic prices $s(t)$. Starting from $t_0$, one measures the number of days until a level $+\rho$, respectively $-\rho$, is reached. The differences $t_{+\rho} - t_0$ and $t_{-\rho} - t_0$ are denoted $\tau_{+\rho}$ and $\tau_{-\rho}$ respectively. Figure adapted from Donangelo [44].

2.1.1 Empirical Results

As introduced in the beginning of this chapter, Simonsen et al. were the first to apply the inverse statistics approach to financial data [38]. Concentrating on financial indices, with the DJIA in particular, Simonsen et al. show that the distribution of waiting times exhibits a sharp maximum followed by a ‘fat tail’ [38]. The maximum of these distributions is the most probable time of producing a return of $\rho$, and has been given the name optimal investment horizon (See figure 2.2).

Figure 2.2: Probability distribution function (pdf), $p(\tau_{\rho})$ of the waiting times $\tau_{\rho}$, with a return level $\rho = 5\%$. The figure is based on the Dow Jones Industrial Average (DJIA). The open circles represent the empirical pdf for the given return level. One can see the optimal investment horizon at approximately $\tau_{\rho}^* = 15$ trading days. The inset is the same distribution on a log-log scale to highlight the power-law behavior of the tale. Finally the solid line represents a maximum likelihood fit to the functional form 2.5. Figure taken from [38].

In the same paper, Simonsen et al. investigate the relationship between the waiting times and the return level $\rho$. If the return level $\rho \to 0^+$, the waiting time probability distribution function is known in the literature as the first return probability distribution for the underlying process [18,60]. Figure 2.3 shows the empirical cumulative distribution $P(\tau_0)$ (solid black curve) for the DJIA over the same period as in figure 2.2. One can observe
2. Theoretical Background

that the tails scale like a power-law $P(\tau_0) \sim \tau_0^{-1/2}$. The value of the exponent is coherent with the assumption that the asset price $S(t)$ can be approximated by a geometrical Brownian motion, as it is common in the financial literature [30,32]. Indeed, if $S(t)$ is assumed to follow a geometrical Brownian, then $s(t) = \ln S(t)$ can be approximated by an ordinary Brownian motion. In such case the literature shows that the first-return probability distribution scales as $p(t) \sim \tau^{H-2}$ [18,60], where $H$ is the Hurst exponent. Since $P(t) \sim \tau^{H-1}$ and the Hurst exponent is equal to $H = 1/2$ for an ordinary Brownian motion, the empirical scaling $P(\tau_0) \sim \tau_0^{-1/2}$ is a consequence of the geometrical Brownian behavior of $S(t)$ [38].

![Figure 2.3](image-url)

**Figure 2.3:** The empirical cumulative probability distributions (solid lines), $P(\tau_\rho)$, vs. horizon $\tau_\rho$ for the DJIA. The dashed line corresponds to the geometrical Brownian motion assumption for the underlying price process. Figure taken from [38].

The cumulative distributions $P(\tau_\rho) \sim \tau_\rho^{\alpha_\rho}$ for other return levels can also be seen in figure 2.3. One can see that the tail exponent $\alpha_\rho$ is relatively insensitive to the return level [38]. Again, one finds $\alpha_\rho \approx 1/2$, consistent with the geometrical Brownian motion behavior. More importantly, Simonsen et al. observed that, as the return level $\rho$ is increased, the most likely horizon moves away from $\tau_0 = 1$ towards larger values: the optimal investment horizon $\tau^*_\rho$ depends on the level of return $\rho$ [38].

As introduced earlier, the time needed for a general time series to reach a certain level of return is known in the literature as the first passage problem, the answer of which is known for a Brownian motion [36,60]:

$$p(t) = \frac{\rho}{\sqrt{4\pi Dt}} \exp \left( - \frac{\rho^2}{4Dt} \right)$$ (2.3)

where $D$ denotes the diffusion constant given by: $D = \sigma^2/2\Delta t$. This form belongs to the family of inverse gamma distributions (also known as Wald
distributions). The relation \( p(t) \sim t^{-3/2} \) can be retrieved by letting \( t \to \infty \). The maximum of this distribution effectively moves away from \( \tau^*_\rho = 1 \) when \( \rho \neq 0 \) \[36, 60\]:

\[
\tau^*_\rho = \frac{\rho^2}{6D} = \frac{\Delta t \rho^2}{3 \sigma^2}
\]

(2.4)

where \( \Delta t \) is the time interval between two consecutive steps (in this paper we use \( \Delta t = 1 \)), \( \sigma \) is the standard deviation of the random process step size distribution.

In order to fit a functional form to the distribution in figure 2.2, Simonsen et al. generalize the above expression and suggest the following function \[38\]:

\[
p(t) = \frac{\nu}{\Gamma\left(\frac{\alpha}{\nu}\right)} \beta^{2a} \left(\frac{t + t_0}{\nu}\right)^{\alpha+1} \exp\left\{ -\left(\frac{\beta t}{t + t_0}\right)^{\nu} \right\}
\]

(2.5)

which reduces to equation 2.3 in the limit when \( \alpha = 1/2, \beta = a, \nu = 1, t_0 = 0 \), since \( \Gamma(1/2) = \sqrt{\pi} \). Although Simonsen et al. seem to be able to obtain good maximum likelihood fits to the generalized gamma distribution outlined here above, they do not claim that \( \tau^*_\rho \) truly follows such a distribution. However, due to its extensive use in the literature, it is reproduced here for consistency. The maximum of the above distribution, the optimal investment horizon, is located at \[40\]:

\[
\tau^*_\rho = \beta^2 \left(\frac{\nu}{(a + 1)\nu}\right)^{1/\nu} - t_0
\]

(2.6)

A relevant property for an investor would be be dependence of the optimal horizon \( \tau^*_\rho \) on the return level \( \rho \). Simonsen et al. measure it empirically and it is reproduced here in figure 2.4. They find that the increase of the optimal horizon occurs in a systematic fashion \[38\]:

\[
\tau^*_\rho \sim \rho^\gamma
\]

(2.7)

with \( \gamma \approx 1.8 \), see figure 2.4. This value is slightly different than \( \gamma = 2 \) which would be expected for a Brownian motion with first passage equation 2.3. It has been later reported that the exponent \( \gamma \) depends on the market investigated \[43\].

Since Simonsen et al. \[38\], empirical inverse statistics of many different financial systems have been performed. These include the NASDAQ and the S&P500 \[43, 45\], Poland \[46, 49\], the Austrian index (ATX) \[49\], the Korean Composite Stock Price Index \[51\], and 40 different indices from various countries \[43\]. In parallel foreign exchange data \[42\] and mutual funds \[50\] have also been subjects of investigation.
2. Theoretical Background

2.2 Stylized Statistical Properties of Asset Returns

Stylized facts can be thought of as a common denominator among the properties observed in studies of different markets and instruments. Although they are usually formulated as qualitative properties of asset returns, stylized facts are so restrictive that it is difficult to display a stochastic process, which possesses all of them. The same difficulties arise when one attempts to reproduce them with a given financial model.

Most of the empirical stylized statistical properties of asset returns are listed and briefly described in the first part of this section. The following two subsections will describe in greater detail the stylized facts of interest in this work: gain/loss asymmetry and the leverage effect.

Absence of autocorrelations

In liquid markets, asset returns typically do not exhibit any autocorrelation above a certain time scale:

\[ C(\tau) = \text{corr}(r_{\Delta t}(t), r_{\Delta t}(t + \tau)) = 0, \text{ for } \tau \geq 15\min \]  

(2.8)

where \( r_{\Delta t}(t) \) is the log return of \( S(t) \) over \( \Delta t \). In other words, one can ignore autocorrelations of asset returns on any time scale superior to 15 minutes, scale at which some microstructure effects come into play. Such a process is called a martingale: knowledge of past events cannot help predict future movements. More specifically:

\[ E[r_{\Delta t}(t + 1) | r_{\Delta t}(t_0), ..., r_{\Delta t}(t)] = r_{\Delta t}(t) \]  

(2.9)

This results mean that the best estimate of the future price is the actual price. A market showing such properties is called an efficient market.
2.2. Stylized Statistical Properties of Asset Returns

Heavy tails

Most of the early empirical research in financial economics focused on modeling the unconditional distribution of returns:

$$F_t(x) = P(r_{\Delta t}(t) \leq x)$$ (2.10)

for which the probability density function (pdf) is defined as its derivative

$$f_t = \frac{dF_t}{dt}.$$ 

In 1963 already, Mandelbrot pointed out that the normal distribution is not sufficient for modeling the marginal distribution of asset returns and their heavy-tailed character [2]. The deviation from the normal distribution is usually measured by the excess kurtosis of the cumulative probability distribution $F_t$:

$$\kappa = \frac{\langle (r_{\Delta t}(t) - \langle r_{\Delta t}(t) \rangle)^4 \rangle}{\sigma(\Delta t)^4} - 3$$ (2.11)

where $\sigma(\Delta t)^2$ is the variance of the log returns $r_{\Delta t}(t)$. Excess kurtosis is defined such that $\kappa = 0$ for the Gaussian distribution. A positive value of $\kappa$ indicates a fat-tail behavior: a slow asymptotic decay of the pdf.

The non-Gaussian character of the distribution of price changes has been repeatedly observed in various market data: $f_t$ tends to be non-Gaussian, sharp-peaked and heavy tailed. However these features are not restrictive enough for a definite choice of the distribution. One can find in the literature many different parametric models [2,5,9,13,23,26], most of which have in common at least four parameters: a location parameter, a scale or volatility parameter, a parameter describing the tail decay and eventually an asymmetry parameter. The choice among these classes is then a matter of analytical and numerical tractability [34].

Aggregational Gaussianity

As one increases the time scale $\Delta t$ over which returns are calculated, their distribution converges towards a normal distribution. This means that the shape of the distribution depends of the time scale chosen [34].

Intermittency & volatility clustering

Intermittency describes the fact that asset returns display, at any time scale, a high degree of variability. This variability is usually quantified through the presence of irregular bursts in time series of a wide variety of volatility estimators. Because these large market movements have a non-negligible probability of occurrence, they cannot be discarded as outliers. In fact their magnitude may be such that they compose a significant fraction of the long-run aggregated returns.
The absence of autocorrelations in return gave some inertia to random walk models of prices in which the returns are considered to be i.i.d random variables. However, the absence of serial correlation is not a sufficient condition to conclude that the increments are independent. Independence is achieved only if in addition any nonlinear function of the returns does not show any autocorrelation. Because some simple nonlinear functions of returns, such as absolute or squared returns, exhibit significant positive autocorrelation, this property does not hold and log prices cannot be considered as random walks. This is a quantitative signature of the well-known phenomenon of volatility clustering: large price variations are more likely to be followed by large price variations. The tendency of high-volatility events to cluster in time is usually measured by the autocorrelation function of the squared returns:

\[ C_2(\tau) = \text{corr}(|r_{\Delta t}(t + \tau)|^2, |r_{\Delta t}(t)|^2) \]  

(2.12)

where \( r_{\Delta t}(t) \) are the log returns of the asset price process. Empirical studies show that this autocorrelation function remains positive over several days and then decays slowly [13,15,16,20,23,30].

**Slow decay of autocorrelation in absolute returns**

Using a similar function as for volatility clustering, one can quantify the autocorrelation of absolute returns:

\[ C_1(\tau) = \text{corr}(|r_{\Delta t}(t + \tau)|, |r_{\Delta t}(t)|) \]  

(2.13)

The autocorrelation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent \( \beta \in [0.2, 0.4] \) [23,24]:

\[ C_1(\tau) \sim \frac{A}{\tau^\beta} \]  

(2.14)

This can be interpreted as a sign of long-range dependence.

**Volume/volatility correlation**

The trading volume is correlated with all measures of volatility. For both stock indexes and individual large stocks, the first-order daily return autocorrelation tends to decline with volume [14,48].

**Asymmetry in time scales**

Coarse-grained measures of volatility predict fine-scale volatility better than the other way round, i.e., information flows from large to small scales. This is consistent with heterogeneous market hypothesis since short-term traders can react to clusters of coarse volatility, while the level of fine volatility does not affect strategies of long-term traders [39].
2.2. Stylized Statistical Properties of Asset Returns

Multifractility

Some of the properties mentioned above (heavy tails, long range dependence) are the result of the multifractal behavior of the price process. A monofractal defines an object, which has identical properties at different time scales. A generalization of monofractal, for which the properties are related, but not necessarily identical, is called a multifractal. Clouds, shorelines or ice stalactites are very popular examples in nature of multifractals. The notion of multifractals was developed by Benoit Mandelbrot and he was one of the first to apply it to financial time series [25]. If we let $S(t)$ be an asset price process, its moments are defined as:

$$M_q(\tau) = E(|\delta_\tau S(t)|^q) = \langle |S(t + \tau) - S(t)|^q \rangle$$  \hspace{1cm} (2.15)

and they can be scaled as:

$$M_q(\tau) = K_q \tau^{\iota_q(\tau)}$$ \hspace{1cm} (2.16)

where $K_q$ is a constant. If $X(t)$ is a monofractal process, the exponents $\iota_q(\tau)$ are independent of $\tau$ and can be represented by $\iota_q = qH$. $H$ is called the Hurst exponent. Because $H$ is a constant, the scaling spectrum is linear. However, if the (generalized) Hurst exponent $H(q)$ is not constant for different orders of the moment $q$, it describes a non-linear scaling spectrum given by:

$$\iota_q = qH(q)$$ \hspace{1cm} (2.17)

and is called a multifractal singularity spectrum.

Time reversal asymmetry

The statistical properties of the process are not identical upon time reversal of the financial time series. The most striking manifestation of this stylized fact is the leverage effect. The leverage effect describes the fact that most measures of future volatility of an asset are negatively correlated with its past returns. See section 2.2.1.

Gain/loss asymmetry

The gain-loss asymmetry refers to the observation that, for stocks or indices, it takes typically longer to gain 5% than to lose 5% [54]. See section 2.2.2.

2.2.1 Leverage Effect

Another important measure of nonlinear dependence in returns is the leverage effect:

$$L(\tau) = \frac{\text{corr}(|r_{\Delta t}(t + \tau)|^2, r_{\Delta t}(t))}{Z}$$  \hspace{1cm} (2.18)
where $Z$ is a chosen normalization and $r_{\Delta t}$ are the log returns of the price process.

The correlation of returns with following squared returns typically starts at a negative maxima for $\tau = 1$ and decays to zero [21, 35]. This suggests that negative returns lead to a rise in future volatility. Interestingly, the effect is asymmetric: $L(\tau) \neq L(-\tau)$, past volatility does not correlate with future price changes. There is evidence of such a negative effect in stock returns and exchange rates, while for interest rates it is not as clear [21].

![Figure 2.5: The Leverage Effect: negative returns lead to a rise in future volatility.](image)

A first attempt to explain this effect was documented by Black (1976) and Christie (1982). They write that a drop in value of a stock (a negative return) decreases the value of equity, and therefore increases the debt-to-equity ratio, or financial leverage. This, in turn, makes stocks riskier and thereby increase their volatility [6, 7]. This is different than the time-varying risk premium theory that states that higher volatility increases the required return on equity, and therefore decreases stock prices [29].

More recently, Bouchaud (2001) conclude that the leverage effect for stocks might not have a very deep economical significance, but can be assigned to a simple ‘retarded’ effect, where the change of prices are calibrated on an exponential moving average of the price rather than on the current asset prices [33].

Furthermore, Bouchaud (2001) shows that the leverage effect is stronger and decays faster in indices than in individual stocks [35]. This might seem confusing at first, because indices are, by definition, an average over stocks. To account for these qualitative differences, the paper details an alternative mechanism for the leverage effect at the index level. The paper presents a one factor leverage model, in which the ‘market factor’ is responsible for the
leverage effect [33]:

\[ \delta S_i(t) = S_i^R(t)[\beta_i \phi(t) + \epsilon_i(t)] \] (2.19)

in which \( \delta S_i(t) \) is the price increment, \( S_i^R(t) \) is a moving average of the past prices, \( \beta_i \) are some time independent stock parameters, \( \phi(t) \) is the market factor common to all stocks. Finally \( \epsilon_i(t) \) are the innovations, or idiosyncrasies, uncorrelated between stocks or to the market common factor \( \phi(t) \).

Using this model, Bouchaud (2001) show that the influence of the market leverage effect on individual stocks is effectively suppressed due to relatively large ratio between the stock volatility and the market volatility [33]. Nonetheless, for stock indices, a specific market panic phenomenon seems to be responsible for the increased leverage effect observed. Interestingly, the simple one factor model shows that the two effects (retardation and panic) are not incompatible and could both be present in individual stocks and stock indices. The relative amplitude of the retardation effect for indices and of the panic effect for individual stocks make them hard to detect however [33].

In this work, to be able to make more meaningful comparisons between heterovolatile assets, the following normalization of the leverage effect function is used:

\[ L(\tau) = \frac{\text{cov}(r_{\Delta}(t+\tau), r_{\Delta}(t))}{(\text{var}(r_{\Delta}(t)))^{3/2}} \] (2.20)

This specification of the leverage effect function has the advantage of being homogeneous, allowing quantitative comparisons between time series. However, using another normalization function should not change the results qualitatively.

### 2.2.2 Gain/Loss Asymmetry

One of the interesting features of financial time series observed by the usage of inverse statistics is the apparent asymmetry between gains and losses. Jensen et al. are the first to show that if one plots the empirical investment horizon distributions of the DJIA for levels of return \( \pm \rho \) (see figure 2.6), there is a higher probability of finding short investment horizons for \(-\rho\) [40] than for \(+\rho\). The downward movements are faster than the upward ones, this was referred to as gain/loss asymmetry [40].

Figure 2.6 shows the empirical investment horizon distributions for \( \rho \pm 5\% \) based on detrended time series of the Dow Jones Industrial Average (DJIA) index. One can easily observe that the maximum of these two distributions, the optimal investment horizons, are shifted one from another. This means it is more probable to find short investment horizons for losses.
2. Theoretical Background

![Figure 2.6: Gain/loss asymmetry: draw-downs are faster than draw-ups. Figure taken from [40].](image)

\( \rho = -5\% \) than for gains \( \rho = 5\% \). The continuous line represents the maximum likelihood fit of the empirical data to the functional form 2.5 described in section 2.1. Jensen et al. find similar results for the S&P500 or the NASDAQ indices [40].

Jensen et al. also investigate the dependence of the optimal investment horizon on the level of return and show that this feature is not unique to \( \rho = 5\% \) (see figure 2.7). Although they find no asymmetry for small level of returns, asymmetry emerges and increases above a certain \( |\rho| \), until it saturates at some constant level [40].

![Figure 2.7: The optimal investment horizon for the DJIA depends on the level of return \(|\rho|\). Recall that for a geometrical Brownian process: \( \tau_p^* \sim \rho^\gamma \), with \( \gamma = 2 \). Figure taken from [40].](image)

Since Jensen et al. [40], the gain/loss asymmetry has been consistently found in mature and liquid markets [45]. Although the gain/loss asymmetry has also been observed in emerging markets [46,49] there is little
consensus on its sign: both positive and negative value of $\tau^+_{+\rho} - \tau^+_{-\rho}$ have been reported in the literature [46,49].

Another particular and interesting empirical result on gain/loss asymmetry is the apparent difference between indices and individual stocks. Johansen et al. compare the empirical investment horizon distributions of the DJIA with the corresponding distributions for a single stocks in the DJIA as well as their average [45]. Although not much can be said on individual stocks due to the poverty of the statistics, the distributions seem to be the same for both positive and negative values of $\rho$. To obtain better statistics and confirm this result, Johansen et al. average separately the gain and loss distributions of the 21 stocks investigated (see figure 2.8). Again, no gain/loss asymmetry is detected at a $\rho = 5\%$ level [45].

The lack of gain/loss asymmetry at the individual component level led to the creation of theories trying to explain this emergent property. Donangelo et al. show with an extremely simple stock synchronization model that collective movements of the overall market can lead to gain/loss asymmetry [44]. Defining individual returns that do not display any asymmetry: $r_i(t) = \ln(S_i(t+1)/S_i(t)) = \epsilon_i(t)\delta$, where $\epsilon_i(t) \pm 1$ is chosen randomly and $\delta > 0$ is the amplitude of the movement. The value of the index is calculated similarly to the DJIA:

$$I(t) = \frac{1}{N} \sum_{i=1}^{N} S_i(t) = \frac{1}{N} \sum_{i=1}^{N} \exp s_i(t)$$

where $s_i(t) = \ln S_i(t)$. The initial values $s_i(0) = s_j(0) \forall i, j$ are chosen arbitrarily. Synchronization is introduced via simultaneous down movements that occur for all stocks with a probability $p$, called fear factor. With a probability $1 - p$ the stocks undergo their independent price process. The
fear factor parameter $p$ is supposed to reflect a collective anxiety state of investors. The results of the model are reproduced in figure 2.9. The model reproduces nicely a similar behavior to the DJIA and shows no asymmetry at the individual stock level (inlet of figure 2.9) [44].

Figure 2.9: The asymmetric synchronous model by Donangelo et al. The inverse statistics obtained within this model for an index of $N=30$ stocks and a level $\rho = 5\sigma$, $\sigma$ denoting the daily volatility of the index. The dashed line represents the result when the fear factor parameter $p = 0$, for which there is no asymmetry. The inset shows the same distributions for the individual stocks (components) of the index. Figure taken from [44].

This model is challenged by Siven et al., who offer a multi scale view on the gain/loss asymmetry in [53]. The paper investigates by the usage of discrete wavelet filtering on what time scale the gain/loss asymmetry emerges. They find that if enough low frequency content (64-128 days) is removed, the gain/loss asymmetry disappears [53]. This indicates that the asymmetry is due to long, correlated, down-movements of prices. The asymmetric synchronous model from Donangelo et al. [44], on the other hand, is based on highly correlated local losses: the synchronization of stocks lasts only one time period.

To remedy the apparent failure of the asymmetric synchronous market model, Siven et al. introduce a generalization of it, in which the market may stay in the distressed state for multiple days [53]. This is established by the parameters $p_{rd}$ and $p_{dr}$ denoting the probabilities of changing from the regular to distressed state and vice versa. In the regular state, all stocks follow independent geometric Brownian motions with drifts $\mu_r$ and standard deviation $\eta$:

$$S_i(t + \Delta t) = S_i(t) \exp(\mu_r - \frac{\xi^2}{2}) \Delta t + \xi \sqrt{\Delta t} Z_i$$  \hspace{1cm} (2.22)
where \( Z_i \sim N(0, 1), i = 1, ..., N \). In the distressed state, all the stocks move together with a negative drift \( \mu_d < 0 \):

\[
S_i(t + \Delta t) = S_i(t) \exp(\mu_d - \frac{\zeta^2}{2})\Delta t + \zeta \sqrt{\Delta t} Z
\]  

(2.23)

where the single random variable \( Z \sim N(0, 1) \) drives the price change in all the stocks. This specification reproduces synthetically the empirically observed gain/loss asymmetry and behaves in the same way when subject to filtering [53].

The temporal dependence structure leading to gain/loss asymmetry has also been investigated. It was shown that scrambling the time series remove completely gain/loss asymmetry in an index [55]. This observation suggests that gain/loss asymmetry is the expression of a temporal dependence structure in the index. Furthermore, Siven et al. construct an artificial equal weight index, made of 12 DJIA constituents, and show that it displays gain/loss asymmetry [55]. Using mean mutual information and mean correlation as measures of dependence, Siven et al. show that there is a greater degree of dependence between constituents in downturns than in upturns [55].

More recently, Ahlgren et al. show that individual stocks actually do show gain/loss asymmetry [56]. Averaging the inverse statistics over 1123 stock time series belonging to mature markets and using a return level of \( \rho = 5\sigma \), they observe that \( \tau^-_{\rho} \sim 10 \) and \( \tau^+_{\rho} \sim 16 \) (see figure 2.10). This apparent contradiction to previous studies may only reflect a detection at a different magnitude: the return level must be measured in relation to the volatility of the price process (for e.g., \( \rho = 5\sigma \)) and not in absolute percentage terms (previous studies used \( \rho = 5\% \)).

As introduced in the previous paragraph, inverse statistics should be compared based on a scaling of the return barrier with the realized volatility of the financial time series. Ahlgren et al. do such a study and use a new unit of time, \( \Delta t(f) \sim |r_{\Delta t}(t)| \), as a function of the original time unit \( \Delta t(t) \). Using this definition, and scaling the time in such a way that the total duration of the time series is preserved, Ahlgren et al. find that gain/loss asymmetry disappears independently of the return level \( \rho \) [56].

The most recent addition to the study of gain/loss asymmetry in financial time series is the perspective of regimes as introduced by Ahlgren et al [56]. Looking on a coarser scale at the DJIA, one can observe different behaviors in the system (see figure 2.11). Ahlgren et al. explain first through a non-mathematical model that analyzing data and observations issued from different regimes can lead to false results and conclusions. They even go one step further by formalizing these observations in a Gaussian
2. Theoretical Background

Figure 2.10: The average of the inverse statistics of 1123 different stocks belonging to mature markets. This figure supports the fact that single stock constituents do possess gain/loss asymmetry. Figure taken from [56].

Mixture model, which is able to reproduce the empirical observations [56]. Essentially, the model shows that a simple autocorrelated structure in state occurrences in sufficient to generate gain/loss asymmetry with processes that do not possess these characteristics. Ahlgren et al. conclude that gain/loss asymmetry could be only an effect of the analysis method alone if multiple data generation mechanisms are present [56].

Figure 2.11: The log prices of the DJIA. The black lines represent coarsely divided regimes. Independently of the scale or index chosen, one can easily find some flat, volatile regimes and some steep calmer ones. Figure taken from [56].
2.2.3 Linking Leverage and Gain/Loss Asymmetry

Because both the leverage effect and gain/loss asymmetry seem to share many common features, Ahlgren et al. attempt to link them together [47] with the Frustration Governed Market model. Although the fear factor model suggested by Donangelo [44] shows that the leverage effect is not necessary to produce gain/loss asymmetry, it is still an open question if both stylized facts are not different measures of the same property [47]. In the model suggested by Ahlgren et al., the main difference with the Fear Factor model is that stock dynamics are controlled through a time-dependent stock volatility, $\sigma(t)$, assumed equivalent for all stocks. All stocks are assumed to follow a geometrical Brownian process with common volatility $\sigma(t)$ and uncorrelated innovations $\varepsilon_i(t) \sim N(0, 1)$ The index is constructed in the same was as in the Fear Factor model:

$$I(t) = \frac{1}{N} \sum_{i=1}^{N} S_i(t)$$

$$S_i(t) = \exp \left( s_i(t) \right) = \exp \left( s_i(t-1) + \varepsilon_i(t)\sigma(t) \right)$$

where $S_i(0) = S_j(0) \forall i, j$. Ahlgren et al. describe that in periods of downward trend in $I(t)$, investors may replace some investments to adapt to the unfavorable situation, depending on their strategy. In other words, investors become frustrated and this leads to an increase in volatility [6]. The effect is opposite when the index is in an upward trend: why should one change a winning strategy? Ahlgren et al. therefore introduce ‘excited frustrated states’ which are gradually relaxed toward a fundamental long-term volatility level $\sigma_0$:

$$\frac{\partial \sigma(t)}{\partial t} = -\frac{(\sigma(t) - \sigma_0)}{\kappa} - A\Theta[-r_\Delta(t)]r_\Delta(t)$$

where $\Theta$ is the Heaviside step-function, $r_\Delta(t)$ the logarithmic return of $I(t)$, $\kappa$ the characteristic volatility decay time and $A$ a positive amplitude. The model is able to reproduce the gain/loss asymmetry and the leverage observed in indices [47]. However, using another set of parameters, it also shows considerable leverage with no discernible gain/loss asymmetry [47].

Siven and Lins investigate the link between the Leverage Effect and Gain/Loss asymmetry assuming that individual stocks do display asymmetry [54]. Using the Exponential GARCH (EGARCH) specification introduced by Nelson in 1991 [12], they are able to show close linkage between the two stylized facts (see figure 2.12). Heuristically they explain that when a stock is close but above the lower barrier $S(t_0) - \rho$, it is likely it just experienced negative returns. If it did, the leverage effect would kick in and increase leverage, thus increasing the probability of crossing the lower barrier. This
probability would be higher than the opposite situation, in which \( S(t) \) is close to the upper barrier \( S(t_0) + \rho \) [54]. They obtain similar results using a retarded volatility model (not reproduced here).

Figure 2.12: Magnitude of the leverage effect (measured by \( A \) in the fit \( L(t) = -A e^{-\tau / T} \)) (left) and the gain/loss asymmetry measured by \( dM = \tau^{*}_\rho - \tau^{*}_{-\rho} \). Both are in function of \( a1a \), the ‘leverage parameter’ in the EGARCH(1,1) model. Figure taken from [54].

In summary, the results from Ahlgren (2007) and Siven (2009) seem to indicate the leverage effect and gain/loss asymmetry are related. None of these papers investigate the differences in amplitude between stocks and indices, nor do they show the effect of correlation on both of these stylized facts. This work tries to explain how stock cross correlation can increase the leverage effect and gain/loss asymmetry in indices.

### 2.3 Modeling of Financial Time Series

The aim of a modeling financial time series is to construct a model for the underlying stochastic process, based on which one can analyze the causal structure of the process or obtain precise predictions. The main difficulties in financial modeling arise from the numerous stylized facts. Most of the statistical regularities presented in the previous sections are difficult to reproduce artificially using stochastic models. Among the many families of financial models available, we chose to focus on the one we use in this work: univariate GARCH models.

Because volatility is considered a measure of risk, for which investors want a premium when investing in risky assets, modeling volatility in asset returns is critical for financial economists. For example, banks and other financial institutions apply value-at-risk models to assess their risks. For these reasons, modeling and forecasting volatility - or the covariance structure of asset returns - is a main challenge in modeling financial time series.

Models of Autoregressive Conditional Heteroskedasticity (ARCH) are a
2.3. Modeling of Financial Time Series

A popular way of parametrizing the higher order dependence in asset returns is a uncorrelated for a lag period over $\tau > 15$ minutes. The same section showed that correlations between higher orders remain possible, most notably the autocorrelation of volatility and the leverage effect (see section 2.2). The family of Autoregressive Conditional Heteroskedastic models (ARCH) offer an interesting framework to integrate both of these stylized facts.

### 2.3.1 ARCH

The first and most basic model of heteroskedasticity is the autoregressive conditional heteroskedasticity model (ARCH). The first ARCH model was introduced by Engle in 1982 and was applied to parametrizing conditional heteroskedasticity in a wage-price equation for the United Kingdom [8].

To answer the questions Engle was facing at the time, he required a model in which uncertainty could change over time. The ARCH model allowed to do so. Let $r_t$ be a random variable that has a mean and a variance conditionally on the information set $\mathcal{F}_{t-1}$ (the $\sigma$-field generated by $r_{t-1}$). The ARCH model of $r_t$ has the following properties:

a. $E[r_t|\mathcal{F}_{t-1}] = 0$

b. The conditional variance $\sigma_t^2 = E[(r_t^2|\mathcal{F}_{t-1})]$ is a nontrivial positive-valued parametric function of $\mathcal{F}_{t-1}$.

Engle assumed that the random variable $r_t$ can be decomposed in terms of independent, identically distributed (iid) random variables $\eta_t$ and the conditional variance $\sigma_t^2$:

$$r_t = \eta_t \sigma_t \tag{2.27}$$

where $\eta_t$ has zero mean and unit variance. This implies $r_t|\mathcal{F}_{t-1} \sim D(0, \sigma_t^2)$ where $D$ stands for the distribution (usually assumed to be a normal or a leptokurtic one). With these conditions set, the following variance defines an ARCH model of order $q$:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 \tag{2.28}$$

where $\alpha_0 > 0, \alpha_j \geq 0, j = 1, ..., q - 1$, and $\alpha_q > 0$. The parameter restrictions in 2.28 form a necessary and sufficient condition for the positivity of the conditional variance. Although the first applications of the ARCH model were not financial ones, the model was quickly recognized for its potential in financial applications that require forecasting of future volatility.

### 2.3.2 GARCH

In current applications the ARCH model specified above is often replaced by the generalized ARCH (GARCH), which was introduced by Bollerslev in...
In this generalization, the conditional variance also depends on its own lags:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$  \hspace{1cm} (2.29)

Compared to equation 2.28, the GARCH specification 2.29 has the property that the unconditional autocorrelation of $r_t^2$ can decay exponentially but slower. In the ARCH model the decay is too rapid compared to what is observed in typical empirical financial time series [52]. This could be solved by using a large lag $q$ in 2.28, but because 2.29 is a more parsimonious model of the conditional variance than a higher-order ARCH model, it is usually preferred.

There are many different specifications and variations in the GARCH family of models. The most applied specification is the GARCH(1,1) model, for which $p = q = 1$ in 2.29. A sufficient condition for the positivity of the conditional variance is satisfied with $\alpha_0 > 0, \alpha_j \geq 0, j = 1, \ldots, q$; $\beta_j \geq 0, j = 1, \ldots, p$. The necessary and sufficient conditions for positivity of the conditional variance in higher-order GARCH models are more complicated and are given in [13], for example. It is important to note that for the GARCH model to be identified when at least one $\beta_j > 0$, one has to require that also at least one $\alpha_j > 0$. If $\alpha_1, \ldots, \alpha_q = 0$, the conditional and unconditional variances of $r_t$ are equal and $\beta_1, \ldots, \beta_p$ are unidentified nuisance parameters. Weak stationarity for the GARCH(p,q) process is guaranteed if $\sum_{j=1}^{q} \alpha_j + \sum_{j=1}^{p} \beta_j < 1$. It is worth noting that the GARCH model is a special case of an infinite-order (ARCH(∞)) model 2.28 with:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j r_{t-j}^2$$  \hspace{1cm} (2.30)

This (ARCH(∞)) representation is useful for the understanding of the existence of moments and long memory [52]. Many other specifications of the GARCH(p,q) model and variations thereof, which are not presented here, can be found for example in [52] or [58].

### 2.3.3 EGARCH

The exponential GARCH model is another popular variation in the ARCH family of models. When first introduced in 1991, Nelson wanted to solve two main criticisms of the standard GARCH model [12]:

- **a.** Parameter restrictions to ensure the positivity of the conditional variance.
- **b.** GARCH does not allow for asymmetric response to shocks.
2.3. Modeling of Financial Time Series

A family of EGARCH\((p,q)\) models may be defined as follows:

\[
\ln \sigma_t^2 = a_0 + \sum_{j=1}^{q} g_j(\eta_{t-j}) + \sum_{j=1}^{p} \beta_j \ln \sigma_{t-j}^2
\]  

(2.31)

where \(g_j(\eta_{t-j}) = a_j \eta_{t-j} + \varphi_j(|\eta_{t-j}| - E|\eta_{t-j}|), j = 1, ..., q\), equation 2.31 becomes the model of Nelson (1991). As can be deduced from its specification, no parameter restrictions are necessary to ensure positivity of \(\sigma_t^2\). Furthermore, the parameters \(a_j, j = 1, ..., q\) make an asymmetric response to shocks possible.

As for the standard GARCH family, the most applied specification in practice is the first-order model, the EGARCH\((1,1)\). Nelson derived conditions for moments of the general infinite-order Exponential ARCH model [12]. Transposed into the EGARCH model shown in 2.31, in which not all \(a_j\) and \(\varphi_j\) equal zero, these existence conditions imply that if the innovations \(\eta_t\) have all moments and \(\sum_{j=1}^{p} \beta_j^2 < 1\), then all moments for the EGARCH process \(r_t\) exist. This is different from the family of GARCH models considered previously, for which the moments conditions become more and more stringent for higher and higher even moments. Expressions for the moments of the first-order EGARCH process can be found in [37]; for the more general case see He (2000) [31].

**EGARCH specification used in this work**

In this work we use the EGARCH\((1,1)\) specification introduced by Nelson in 1991 [12], for which the conditional variance can be described as:

\[
\log(\sigma_t^2) = a_0 + a_{1a} \frac{\varepsilon_{t-1}}{\sigma_{t-1}} + a_{1b} \left( \frac{|\varepsilon_{t-1}|}{\sigma_{t-1}} - E \left[ \frac{|\varepsilon_{t-1}|}{\sigma_{t-1}} \right] \right) + b_1 \log \sigma_{t-1}^2
\]  

(2.32)

in which \(a_0, a_{1a}, a_{1b}, b_1\) are parameters, \(\sigma_{t-1}\) and \(\varepsilon_{t-1}\) are the conditional variance and the innovations of the previous time step, respectively. The parameter \(a_{1a}\) controls the effect of the sign of the innovations on the conditional variance, whereas the parameter \(a_{1b}\) controls the impact of their magnitude. Because \(\varepsilon_t \sim \mathcal{N}(0, \sigma_t^2)\), the variable \(\frac{\varepsilon_t}{\sigma_t}\) follows a standard normal distribution, and we have [3]:

\[
E \left[ \frac{|\varepsilon_t|}{\sigma_t} \right] = \sqrt{\frac{2}{\pi}}
\]  

(2.33)

In the base model used in this paper, the log returns of the asset are represented as:

\[
r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_t^2)
\]  

(2.34)

where \(r_t\) represents the log returns of a given asset, with expectation \(\mu\). The innovations at every time step \(\varepsilon_t\) follow a normal distribution with standard deviation \(\sigma_t\).
2. Theoretical Background

For the EGARCH(1,1) model presented here the average expected volatility \( \bar{\sigma}^2 \) can be calculated as:

\[
\bar{\sigma}^2 = \exp \left( \frac{a_0 - a_{1b} \sqrt{2/\pi}}{1 - b_1} + \frac{1}{2} \left( a_{1d}^2 + a_{1b}^2 \right) \right) \\
\prod_{m=0}^{\infty} \left[ F_m(a_{1d}, a_{1b}, b_1) + F_m(-a_{1d}, a_{1b}, b_1) \right] 
\]

(2.35)

where \( F_m \) is defined as:

\[
F_m = N \left( b_m (a_{1b} - a_{1d}) \right) \exp(b_m^2 a_{1d} a_{1b}) 
\]

(2.36)

in which \( N[. \] is the cumulative standard normal distribution. We refer the reader to Appendix B of [17] for a detailed proof, and [22] equation (6) for a simplified expression of the result.
Chapter 3

Gain/Loss Asymmetry and the Leverage Effect

The existing literature has not yet reached a consensus on the link between the leverage effect and gain/loss asymmetry, nor on its amplification at an index or portfolio level. In this master’s thesis, we analyze gain/loss asymmetry under three different points of view. First, we test empirically the results described in the literature, without any prior data treatment. In parallel we investigate the relation between the inverse statistics and the leverage of equity indices compared to those of their constituents. In a second part, we introduce a model to generate financial time series and study their behavior when they are treated as part of an index. Finally, we assess how different variations of the leverage effect may impact gain/loss asymmetry.

The first section concentrates on empirics and tests some of the results found in the literature. We start by briefly reviewing the generalized inverse gamma distribution and assess whether it can fit the empirical distributions of the inverse statistics. The difference in the leverage effect and gain/loss asymmetry between stocks and equity indices is measured in the subsequent section. Whereas gain/loss asymmetry and the leverage effect have both been observed independently in the existing literature, here we study both of them in parallel in the aim of investigating their possible relation. In the third subsection we build two different artificial equity indices, based on which we can investigate the differences between the properties of the index and its constituents.

In the second section, we generate synthetic financial time series to test the impact of asset cross-correlation on the amplification of the leverage effect and gain/loss asymmetry. A common factor or market model shows that it can nicely reproduce the behavior observed in the empirical sections of this work. For sanity checks, the next subsection proves that an inverse model is
3. Gain/Loss Asymmetry and the Leverage Effect

not able to reproduce those features. Finally, we investigate the dynamics of convergence towards the common factor in terms of the number of assets in a portfolio or index.

The last section aims at gaining a broader understanding of the impact of the leverage effect on gain/loss asymmetry. First, different synthetic series bearing different time decays of the leverage effect are generated and the resulting gain/loss asymmetry is compared. The last subsection aims at explaining what kind of mechanisms are necessary to generate the missing features of the synthetic common factor model introduced previously.

3.1 Empirics

3.1.1 Generalized Inverse Gamma Fit

Fitting the generalized inverse Gamma distribution to the empirical inverse pdfs is tedious and yields unsatisfactory results. As introduced in the theoretical section of this work, Simonsen et al. suggest the usage of the functional form 2.5 to fit the empirical distributions of waiting times [38]. This generalized inverse gamma distribution has the advantage of being extremely flexible, but the various values of the fitted parameters are difficult to interpret. Furthermore, as we tried to fit this functional form with the maximum likelihood method, the results were largely unsatisfactory. We were unable to reproduce the fits displayed by Simonsen et al. [38]: the maximum of the distribution is biased towards larger values and the parameters are very far off what was written in the literature (data not shown). The profiles of the maximum likelihood function are extremely flat and often fail to converge. For these reasons, using the expression [40]:

$$\tau^*_\rho = \beta^2 \left( \frac{\nu}{\nu + 1} \right)^{1/\nu} - t_0$$

fails to capture the optimal investment horizon of the empirical distributions. In this work we prefer using the mode of the empirical distribution, or the average of the three most probable values when the absolute maximum is ambiguous.

Together, these results suggest that the inverse generalized gamma distribution may not be the best candidate to fitting the distributions of waiting times. Perhaps the distribution displayed arises from different mechanisms and is the result of two or more superposed functional forms.

3.1.2 Stocks VS Indices

In this first subsection, we investigate the differences in gain/loss asymmetry and in the leverage effect between stocks and indices. To assess these
differences, while holding all other parameters equal, we compute and average the inverse statistics of four different indices and 457 stocks issued out of the S&P500 over a 20 year period (Jan 1990 - Dec 2010). We do not perform any detrending, which may affect strongly the results. The list of the indices and stocks used are listed in Appendix A. Because of the availability of individual stock data over the period investigated this subset of stocks is chosen over the actual 500 constituents of the S&P500. Using the S&P500 alone against its constituents over the same period would not change the results found in this subsection, however they would be much more noisy (data not shown).

Inverse statistics of four different indices show that they have stronger gain/loss asymmetry than individual stocks over the same period. The distributions of waiting times for $\rho = \pm 5\sigma$ are shown in figure 3.1. The figure shows a clear asymmetry between gains and losses for the indices (red markers), this asymmetry is much smaller for the individual stocks (blue markers). In both cases the level of return used is $\rho = 5\sigma$. The optimal investment horizons are further one from another in the case of the indices: it takes longer to make gains of $\rho = 5\sigma$, while it is faster to lose $\rho = -5\sigma$! These results for indices are similar to what has been found in the literature so far [40, 45, 46, 49]. The asymmetry displayed by stocks in this figure also confirm what has been shown by Ahlgren et al. [56]. Interestingly, the frequency of the optimal investment for losses in the index is relatively high compared to the three other distributions (all distributions are normalized such that they integrate to 1).

To compare the importance of the gain/loss asymmetry across different
3. Gain/Loss Asymmetry and the Leverage Effect

assets independently of the return level \( \rho \) chosen, we introduce the following asymmetry ratio:

\[
\text{Asymmetry ratio}, \ r_\rho = \frac{\tau_{+\rho} - \tau_{-\rho}}{\tau_{-\rho}}
\]  

(3.2)

For the indices and stocks investigated in this subsection, the asymmetry ratio of the average inverse statistics is \( \simeq 0.85 \) and \( \simeq 0.25 \) for the indices and stocks respectively. The locations of the optimal investment horizons and the asymmetry ratios are listed in table 3.1. The optimal investment horizons \( \tau^* \) are estimated by the mode of the distribution of waiting times. In case of bad statistics, or ambiguous maxima, we average the positions of the three highest frequencies.

| Indices  | \( \tau_{+\rho} \) | \( \tau_{-\rho} \slash \tau_{+\rho} \) | \( \max |L(\tau)| \) | \( \langle \sigma \rangle \) |
|-----------|-----------------|----------------------------|-----------------|--------|
| Indices   | 9               | 16.67                      | 0.8522          | 0.3309 | 0.0137 |
| Stocks    | 12              | 15                         | 0.25            | 0.2942 | 0.0253 |

Table 3.1: Gain/loss asymmetry for the indices and stocks listed in appendix A.

The maximum amplitude of the leverage effect is stronger in indices than in stocks. As introduced in subsection 2.2.3, Ahlgren et al. show that the leverage effect and gain/loss asymmetry share many common features [47]. Furthermore, recall that Siven and Lins show a linear relationship between the leverage parameter in the EGARCH(1,1) and gain/loss asymmetry [54]. Naturally, in an attempt to provide an explanation for the observed results described above, we compute here the average leverage effect for the four stocks and 457 stocks investigated. No detrending is performed, and the adjusted daily closing prices are used. According to the previous observations, the larger asymmetry displayed in indices could be the effect of a larger underlying leverage effect. Figure 3.2 shows the average leverage effects for the two classes of financial time series investigated, using the same level of return and time period as above. The maximum amplitudes of the leverage effect are listed in table 3.1. As found previously by Bouchaud et al., the leverage effect is larger for indices than for stocks [33]. The maximum of the curves are located at \( \max |L(\tau)| = -0.3309 \) and \( \max |L(\tau)| = -0.2942 \) for indices and stocks respectively. Thus, indices display a higher maximum absolute correlation between past negative returns and future volatility, independently from the leverage function used. Recall from subsection 2.2.1 that Bouchaud [33] uses an inhomogeneous measure of the leverage effect, which is strongly dependent on the volatility of the underlying processes. The average standard deviations of the log-returns for the stocks and indices analyzed are presented in table 3.1. Note that the average index standard deviation is nearly twice lower than the one of the individual stocks.
3.1. Empirics

Figure 3.2: Leverage effect for the indices (red) and stocks (blue) listed in appendix A. The maxima of the absolute values of the leverage effect are located at \( \max |L(\tau)| = 0.3309 \) and \( \max |L(\tau)| = 0.2942 \), for indices and the stocks respectively. It is also worth noting that the maximum is located at \( \tau = 1 \) for the indices. However for the stocks the maximum is already present at \( \tau = 0 \).

Another interesting fact is the positive correlation between past volatility and present returns. Both for stocks and for indices, the leverage effect is positive for \( \tau = -1 \). This would imply that past volatility is correlated with positive returns. This feature was already noted by Bouchaud et al. [33]. Their explanation was that large daily drops, which increase volatility, are often followed by rebound days. This could mean that some price drops are often exaggerated and reach undervalued levels. Gains are made in the following days when the asset price readjusts itself to its intrinsic value.

Although the leverage effect initially decays faster for indices than for stocks, the effect is more persistent over time. One can observe in figure 3.2 that the absolute value of the leverage effect is consistently higher for indices (red curve) than for stocks (blue curve). If we follow Bouchaud et al. [33] and compute exponential fits to the leverage effect curves, we observe that the decay is faster for indices than for stocks (see left plot in figure B.1). The decay time for stocks is very close to those observed by Bouchaud et al. (about 40 days). For the indices on the other hand, we find something a decay time of about 30 days. This time decay is much more similar to the one of stocks than the 10 days observed by Bouchaud et al. Because of the poor fit of the exponential function for \( \tau \) close to 0 we suggest a power law fit of the form \( L(\tau) = A \tau^{-\alpha} \). This form is similar to that observed for the volatility correlation function [27]. As seen on the right plot in figure B.1, the functional form suggested fits very well the leverage function for individual stocks. The high noise in the average leverage function for indices hampers any good fits, recall that 457 stocks are averaged against only 4 indices. We
3. Gain/Loss Asymmetry and the Leverage Effect

conclude that, given the poor statistics, the power law exponents are rather close and that the longer persistence observed in indices is due to the higher maxima.

The maximum of the absolute value of leverage function is delayed for indices. Bouchaud et al. [33] observed that the maxima of the leverage function for stocks and indices are located at $\tau = 0$. Figure 3.2 shows that for indices this is not the case. The maxima of the absolute value of the leverage effect are located at $\tau = 0$, $\max |L(\tau)| = 0.2942$ and $\tau = 1$, $\max |L(\tau)| = 0.3309$ for stocks and indices respectively. This suggests that the leverage effect in stocks and indices might not have the same underlying mechanisms: in indices the effect is stronger, lasts longer, but starts with an delay of one trading day.

Conclusion

Gain/loss asymmetry and the leverage effect are amplified in indices in comparison to individual stocks. Let us recap the different features observed in this section:

- a. Indices show stronger gain/loss asymmetry than individual stocks.
- b. The maximum of the leverage effect is stronger in indices.
- c. There is a rebound effect in both stocks and indices.
- d. The leverage effect decays faster over time in indices.
- e. The maximum amplitude of the leverage effect starts at $\tau = 1$ for indices and $\tau = 0$ for individual stocks.
3.1. Empirics

Recall we didn’t perform any detrending on the time series, which may affect the probability of the waiting times and may lead to qualitatively and quantitatively different results. We always chose a level of return proportional to the volatility of the time series ($\rho = 5\sigma$), in order to compare stocks and indices that are heterovolatile.

These results confirm the link between the leverage effect and gain/loss asymmetry as shown by Siven and Lins in [54]. Indeed, it seems as stronger gain/loss asymmetry observed empirically in indices is correlated with a larger leverage effect. The explanation offered by Siven and Lins was that when a stock is close but above the lower barrier $S(t_0) - \rho$, it is likely it just experienced negative returns. If it did, the leverage effect would kick in and increase leverage, thus increasing the probability of crossing the lower barrier. This probability would be higher than the opposite situation, in which $S(t)$ is close to the upper barrier $S(t_0) + \rho$. To our knowledge, this is the first time an empirical correlation between the leverage effect and gain/loss asymmetry in stocks and indices is reported. This may indicated that because stocks in an index possess some cross correlation, the leverage effect is amplified and leads to a stronger gain/loss asymmetry.

Our intuition is that the stronger the cross-correlation between assets in a portfolio, the stronger the gain/loss asymmetry the portfolio will display. This hypothesis will be tested in the remainder of this work.

The rebound effect, like the leverage effect, is a property which is amplified in indices but which is already present at the individual stock level. This amplification suggests both of these properties are dependent of asset cross-correlation. Similarly, the faster decay of the leverage effect in indices in comparison to stocks may be due to the fact asset cross-correlation shows period of high correlation and some periods of lower correlation. Consequently, the leverage effect is amplified so long the assets remain correlated strongly enough. Over time, the financial time series of the asset may diverge, leading to a faster decrease of the leverage effect in indices.

The difference in the onset of the leverage effect in stocks and in indices may indicated that stocks possess a property that disappears in the index. The maximum of the leverage effect is located at $\tau = 0$ for individual stocks. This contemporaneous negative correlation between asset returns and volatility is actually different from the leverage effect. The latter explains that an decrease in the stock price increases volatility. In this case however, the effect occurs simultaneously ($\tau = 0$). Literally, this means that negative returns are correlated with larger volatility. Because we are observing closing prices, we only capture part of the dynamics of the stock market. Perhaps negative intraday returns create an intraday panic/herding effect and trigger stop loss strategies, which lead to lower closing prices than upwards moves would. This would mean the individual asset returns
are negatively skewed but when aggregated in an index the effect in not persistent. Another explanation would be that asset returns are symmetrical around a negative expectation (trend or bias) leading to a contemporaneous correlation between negative returns and larger volatility. Both of these features are also explored further in this work.

3.1.3 Artificial Equity Index

To investigate the relationships between the statistics of the indices to those of their constituents an artificial index is created from ten US stocks. A similar experiment was carried out by Siven et al. on twelve stocks, in which they observe gain/loss asymmetry and study the cross-correlations of the constituents (see section 2.2.2 on p. 19). Using the time series of ten different US stocks from Jan 1980 to Dec 2010 (stock details in appendix C), we build artificial indices and compare their inverse statistics to those of their constituents. We use the twelve stocks studied by Siven, but without General Motors, the US subsidiary of which filed for bankruptcy in 2009 nor JPMorgan, which merged with Chase Manhattan Corporation in 2010.

Price Weighted Artificial Index

In this section we build an price weighted index, adjusted for the initial asset price. If we let $I(t)$ be the artificial index time series and $S_i(t)$ the individual asset prices:

$$I(t) = \frac{1}{N} \sum_{i}^{N} \frac{S_i(t)}{S_i(0)} \quad (3.3)$$

This value of the index is not rebalanced at any time thereafter. It behaves like a portfolio with an equal initial investment in the eleven stocks listed in appendix C. This means that a stock which performs better than the others gains importance in the calculation of the index value.

The ten stocks show little gain/loss asymmetry, whereas the effect is larger for indices. As shown on figure 3.4, the optimal investment horizons are located at $\tau^+_{\rho} = 15, \tau^-_{\rho} = 12.667$ for the stocks and at $\tau^+_{\rho} = 18, \tau^-_{\rho} = 12$ for the indices. In both cases the level of return used is $\rho = 5\sigma$. Unfortunately, the statistics for the index are rather noisy and do not give room for any precise interpretations. Nonetheless, the increase in the asymmetry ratio from $r^+_{\rho} \approx 0.185$ to $r^-_{\rho} = 0.5$ is quite large and shows that simply creating an index from the constituents leads to an increase in gain/loss asymmetry. Siven explained this effect by increased correlation between the assets in downturns compared to upturns, thus leading to faster losses [55]. But when we have a look at the positions of the optimal investment horizons, we observe that $\tau^+_{\rho}$ did not move as much as $\tau^-_{\rho}$. The increase in the asymmetry ratio is mostly due to the optimal investment
3.1. Empirics

Figure 3.4: Inverse statistics for 10 US stocks (left) and a price weighted artificial index made of the same stocks (right). Although the statistics are noisy for the artificial index, one can see a slight increase in gain/loss asymmetry: the optimal investment horizon for gains shifts away from $\tau^*_p = 15$ to $\tau^*_p = 18$. The optimal investment horizon for losses decreases slightly from $\tau^*_\rho = 12.667$ to $\tau^*_\rho = 12$. The asymmetry ratio increases from $r_\rho \approx 0.185$ to $r_\rho = 0.5$.

The maximum of the leverage effect seen in the stocks increases drastically when it is measured in the index. The average leverage effect of the stocks and the leverage effect for the index as a whole are shown in figure 3.5. One can see that the maximum of the leverage effect is twice larger in the index.
than in the average of the stocks. The maxima are both located at \( \tau = 0 \). The rebound effect can also be observed here and is amplified in the index.

There is no clear difference in the decay behavior between the stocks and their common artificial index. To evaluate the decay of the function over time, the same fit procedure as in the previous subsection is applied. The results are presented in figure 3.5. Both the stocks and the artificial index seem to share common characteristics regarding their fits. In the exponential fit, the time decay is about 24 days for both the stocks and the index. The amplitude, \( A \), of the function \( A \exp(\tau/T) \) is very similar as well: \( A = 0.186 \) and \( A = 0.208 \) for stocks and the index respectively. Similar results are obtained with the power law fit \( c x^{-\alpha} \): the amplitude \( c \) and exponent \( \alpha \) are nearly identical.

### Arithmetic Average Equally Weighted Artificial Index

Another type of index uses equal weighted asset prices, in which all stocks have the same impact on the index price. If we let \( I(t) \) be the artificial index time series and \( S_i(t) \) the individual asset prices:

\[
I(t) = I(t-1) \frac{1}{N} \sum_{i=1}^{N} \frac{S_i(t)}{S_i(t-1)}
\]  

(3.4)

In this type of index, the asset with the largest percentage change has the greatest impact on the index value.

![Figure 3.6: Inverse statistics for 10 US stocks (left) and an arithmetic average equally weighted artificial index made of the same stocks (right). Although the statistics are noisy for the artificial index, one can see an increase in gain/loss asymmetry: the optimal investment horizons for gains and losses move from: \( \tau^+_{\rho} = 15 \rightarrow 14 \) and \( \tau^-_{\rho} = 12.667 \rightarrow 10 \) for gains and losses respectively. This leads to an increase in the asymmetry ratio from \( r_{\rho} = 0.185 \rightarrow 0.4 \).](image)

When using an arithmetic average equally weighted index, the gain/loss asymmetry increases as well. As shown on figure 3.6, the optimal investment horizons for the index are located at \( \tau^+_{\rho} = 11, \tau^-_{\rho} = 10 \) (recall
that they are located at $\tau^*_p = 15, \tau^*_=- = 12.667$ for the stocks. The level of return used is $\rho = 5\sigma$. Again, the statistics for the index are not extremely precise, but they allow however to see a decrease in the asymmetry ratio from $r_\rho \approx 0.185$ to $r_\rho = 0.1$. This result goes in the other direction than what was observed for the equally weighted index.

![Graph](image)

**Figure 3.7:** Leverage effect for stocks and its artificial equally weighted index. The two exponential fits are plotted on the left graph (same plot as in figure 3.5). The right graph shows power-law fits to the leverage function.

The maximum of the leverage effect seen in the stocks also increases in the equally weighted index. The average leverage effect of the stocks and the leverage effect for the index as a whole are shown in figure 3.7. One can see that the maximum of the leverage effect is also twice larger in the index than in the average of the stocks. The maxima are both located at $\tau = 0$. The rebound effect can also be observed here and is amplified in the index.

As for the price weighted index, there is no clear difference in the decay behavior between the stocks and their common artificial equally weighted index. To evaluate the decay of the function over time, the same fit procedure as in the previous subsection is applied. The results are presented in figure 3.7. The results are very similar to the ones showed in the previous paragraphs. For the idea the decay in the exponential fit dropped slightly from 24.445 days to 22.55, becoming faster than for the stocks. The amplitude, $A$, of the function $A \exp(\tau/T)$, on the other hand, increased: $A = 0.186$ and $A = 0.276$ for stocks and the index respectively. Similar results are obtained with the power law fit $c x^{-a}$: although the amplitude $c$ is slightly larger, $c = -0.449$ and $c = -0.665$ for stocks and the index respectively.
respectively, the exponents $a$ are nearly identical.

**Conclusion**

Creating an artificial index based on empirical financial time series underlines the importance of asset cross-correlation in the amplification of the inverse statistics in an index. By building these two different artificial indices, we can directly compare an index behavior to that of its constituents, all else equal. As expected, the leverage effect is stronger in the index than on the average of the individual stocks. This in turn may lead to the larger gain/loss asymmetry observed in the artificial index. The main difference between the leverage effect and the inverse statistics measured in the index, as opposed to the average if the individual stocks, is the asset cross-correlation. Recall that gain/loss asymmetry disappears altogether if the temporal structure is removed [55]. Measuring inverse statistics after summing the constituents up in an index amplifies the features they have in common and which are synchronized. If the assets were totally uncorrelated, creating an artificial index out of them would sum up to white noise. The origin of the leverage effect and the gain/loss asymmetry in stocks and in indices may be two different mechanisms, as suggested by Bouchaud [33]. Or, its origin may come from a common market factor, which is amplified when measured in an index because of asset cross-correlation. In the following section we investigate a common factor model with stock cross-correlation and how it builds up to the inverse statistics we observe in equity indices.

On the other hand, the maximum of the leverage function does not seem to account entirely for the gain/loss asymmetry observed. As shown in figure 2.12, Siven and Lins show through the leverage parameter of Nelson’s EGARCH specification that there is a nice relation between the amplitude of the leverage effect and gain/loss asymmetry. However, if we compare the results of this subsection with those of the previous one, we do not seem to be able to explain entirely the gain/loss asymmetry by the amplitude of the leverage effect. In the previous subsection, the difference in the maximum of the leverage effect between stocks and indices is rather small (see table 3.1), but in the same time the gain/loss asymmetry is increased much more significantly: $r_\rho = 0.25 \rightarrow 0.8522$. In this subsection on the other hand we have a twofold increase in the leverage effect that leads to a smaller increase in the asymmetry ratio: $r_\rho = 0.25 \rightarrow 0.5$. The difference between the two cases may reside in the time decay of the leverage function. Indeed, the gain/loss asymmetry is larger where the correlation between past negative returns and future volatility is more persistent. In this section the differences were almost nonexistent, leading to a smaller increase in the gain/loss asymmetry. The impact of the time decay of the leverage effect is also tested through the stock model developed in the following section.
3.2 A Simple Model for Stock Correlation

To investigate how the leverage effect and gain/loss asymmetry behave one with another, we build different specifications of EGARCH-based models to simulate asset price processes. Then we build an artificial index composed of these assets and compare their individual properties and inverse statistics. As introduced in section 2.3, the EGARCH specification introduced by Nelson in 1991 [12] offers an ideal framework to simulate asymmetric responses to asset returns.

The following subsections discuss different simulations, which aim at investigating a question of interest. First, we investigate the central hypothesis of this work: asset cross-correlation around a common market factor leads to an amplification of the leverage effect and gain/loss asymmetry. The impact of the index size is investigated in the same subsection. Then we set leverage at the individual level and verify that it does not lead to an amplification when measured in the index in an ‘inverse model’. Finally, we assess the dynamics of convergence towards the properties of the common market factor as a function of the number of assets in the index.

3.2.1 Market Model

To assess the behavior of equity portfolio’s inverse statistics in function of the asset cross-correlation, we build a simple EGARCH based model. We fix correlation and verify in the model that the asymmetry of the portfolio is larger than the individual asymmetry in each asset. Let \( X(t) \) be the log returns of a common factor (market) based on an EGARCH process such as introduced in section 2.3, and \( Y_i(t), i = 1, ..., N \) are the log returns of the \( N \) assets of the portfolio:

\[
Y_i(t) = aX(t) + \xi_i(t), \quad i = 1, ..., N \quad (3.5)
\]

\[
X(t) = \varepsilon(t), \quad \varepsilon(t) \sim \mathcal{N}(0, \sigma^2_{x,t}) \quad (3.6)
\]

\[
\log(\sigma^2_{x,t}) = a_0 + a_1 a \frac{\varepsilon_{t-1}}{\sigma_{x,t-1}} + a_1 b \left( \frac{|\varepsilon_{t-1}|}{\sigma_{x,t-1}} - E \left[ \frac{|\varepsilon_{t-1}|}{\sigma_{x,t-1}} \right] \right) + b_1 \log \sigma^2_{x,t-1} \quad (3.7)
\]

where \( \xi_i(t) \sim \mathcal{N}(0, \sigma^2_i), i = 1, ..., N \) are independent and identically distributed random variables with standard deviation \( \sigma \). Finally, we generate an \( N \) asset, price-weighted portfolio and compare its properties to those of the individual assets. If we let \( I(t) \) be the log returns of this portfolio, we obtain:

\[
I(t) = \frac{1}{N} \sum_{i=1}^{N} Y_i(t) \quad (3.8)
\]
3. GAIN/LOSS ASYMMETRY AND THE LEVERAGE EFFECT

The parameter $\alpha$ allows to control the correlation between the assets $Y_i$. The correlation between $Y_i$ and $Y_j$ is given by (details in Appendix D):

$$\text{corr}(Y_i, Y_j) = \frac{E[Y_i Y_j]}{\sqrt{E[Y_i^2]E[Y_j^2]}} = \frac{\alpha^2 E[X^2]}{\alpha^2 E[X^2] + \sigma_y^2} = \frac{\alpha^2}{\alpha^2 + \frac{\sigma_y^2}{E[X^2]}}$$ (3.9)

Let us denote $\sigma_x^2$ the unconditional variance of the EGARCH returns $X(t)$, such as defined in 2.3. If we chose the EGARCH parameters and $\sigma_y$, such that $E[X^2] = \sigma_x^2$ and $\sigma_y^2$ equal each other, we obtain the simplified form:

$$\text{corr}(Y_i, Y_j) = \frac{\alpha^2}{\alpha^2 + 1}$$ (3.10)

Thus we have a model that incorporates leverage in $X(t)$ and with which we can generate N assets that have a given cross-correlation.

The model can reproduce the increased asymmetry and leverage effect observed in indices compared to stocks. In the simulations shown in figure 3.8 we take the same EGARCH parameters as Siven and Lins, which come from estimates of German stocks by Schmitt et al. [22] (see caption of figure 3.8). From the N assets, an equally weighted index is built. The inverse statistics and the leverage effect are measured at both the individual level and at the index level. In both cases the level of return used is $\rho = 5\sigma$.

The resulting index has optimal investment horizons located at $\tau_{-\rho}^- = 9$ and $\tau_{+\rho}^+ = 16$, whereas the average of those of the stocks are located at 11 days for gain and losses (see figure 3.8). These optimal investment horizons correspond to asymmetry ratios of $r_\rho = 0.78$ and $r_\rho = 0$ for the index and the stocks respectively. The absolute value of the leverage function decays over time exponentially, as expected from the specification of the EGARCH model used (see appendix D). The maximum of the leverage effect
3.2. A Simple Model for Stock Correlation

is located at $\tau = 1$, and very similar to the behavior of indices as denoted in section 3.1.2. Finally, the rebound effect observed in subsection 3.1.2 is not significant here and decreases when measured in the index.

![Figure 3.9: The maximum of the leverage effect also moves away from zero to reach the value of the underlying EGARCH process $\max L(\tau) = 0.45$. This value corresponds to the sum of the absolute values of the EGARCH parameters $a_1a$ and $a_1b$: $a_1a = -0.3, a_1b = 0.15$](image)

Asset synchronization leads to an increase in the amplitude of the leverage effect. To assess the effect of asset correlation in the index we repeat the procedure outlined in the previous paragraph with different values of the parameter $\alpha$. For each value of $\alpha$ we run 1000 realizations and record the average inverse statistics and leverage effect. As asset correlation is increased, the maximum amplitude of the leverage effect for the stocks and for the index converge to that of the underlying EGARCH common process $X(t)$. This maximum value can be estimated from the parameters: it is approximately equal to the sum of the absolute values of the parameters $a_1a$ and $a_1b$. Here again, the dynamics of convergence between the index and the average of the stocks is remarkable: the stocks exhibit a monotonous convex curve between $\max L(\tau)$, whereas the same curve is concave for the index. Half of the maximum amplitude is already reached for asset correlation values of $\approx 0.15$. The same amplitude is reached in the individual stocks only if the asset correlation reaches 0.6, which is four times larger. With this specification however, the model is unable to reproduce the rebound effect observed in the empirical data.

Asset synchronization leads to an increase in gain/loss asymmetry. Another characteristic measured during the simulation outlined in this subsection is the location of the optimal investment horizons. The EGARCH parameters used are the ones outlined in the caption of figure 3.8, the index is composed of 10 assets and the return level is $\rho = 5\sigma$. As it can be seen
3. GAIN/LOSS ASYMMETRY AND THE LEVERAGE EFFECT

Figure 3.10: Changing $\alpha$ from 0 to 1000 shows the effect of asset cross-correlation on the optimal investment horizons. $\tau_{+}^\rho$ drifts from 11 to 27 days, whereas $\tau_{-}^\rho$ slips from 11 to 8 days.

In figure 3.10, increasing the cross-correlation between the assets increases the asymmetry measured in the index faster than in the individual stocks. Therefore at a given correlation, the gain/loss asymmetry is stronger in the index than in the individual stocks, until both converge for large values of correlation with the common factor $X(t)$. Whereas the optimal investment horizon for gains in the index drifts away from $\tau_{+}^\rho = 11$ to reach $\tau_{+}^\rho = 27$, the investment horizon for index losses shows much less variation. $\tau_{-}^\rho$ starts at 11 as well and reaches slowly $\tau_{-}^\rho = 8$. The optimal investment horizons in the stocks have the same border values but the dynamic in between is very different. These border values are no surprise: when the assets are uncorrelated, they are pure geometrical Brownian motions and therefore the index does not exhibit any asymmetry. However, as the correlation is increased, the stocks reach the characteristics of the underlying common returns $X(t)$.

Increasing the number of assets in the index leads to a strong convergence with the common factor. In the two previous subsections, the number of assets was held fixed ($N = 10$). In this subsection we evaluate the impact of varying the number of assets in the index or portfolio, holding the cross-correlation constant ($\alpha = 0.5$, correlation $\sim 0.2$). Figure 3.11 shows the effect of the number of assets on the optimal investment horizons and on the maximum leverage effect observed. Note that the x-axis is logarithmic to show the sigmoid behavior of the curves. The optimal investment horizons converge very fast and in a sigmoid way to the values observed in $X(t)$ (see figure 3.11 (left)). Likewise, the maximum leverage effect in the index is strongly dependent on the number of assets. 10 assets are
3.2. A Simple Model for Stock Correlation

Figure 3.11: Convergence of the index properties to those of the underlying time serie $X(t)$ with increasing number of constituents. The x-axis is in logarithmic scaling to appreciate the dynamics for the values with few assets. At 100 constituents, both properties seem to reach full convergence.

sufficient to reach half of the maximum leverage effect of $X(t)$, whereas at 100 assets convergence is total (see figure 3.11 (right)). These results show that we should compare gain/loss asymmetry in indices only between those that have the same number of assets, at least for indices of less than 100 constituents. For example, the SMI or the DJIA only have 20 and 30 stocks respectively, but the S&P500 has 500 of them, the Russell family of stock indices up to 3000-4000. Looking back at subsection 3.1.3, three of the four indices had over 100 assets. Only the DJIA had 30 constituents. However, repeating the procedure without the DJIA shows qualitatively similar results with more noise (not shown).

Conclusion

The market model introduced in this section is able to reproduce most of the properties observed empirically and resembles the capital asset pricing model. The assets are all based on a common factor, which we can call the market in analogy to the well-known capital asset pricing model (CAPM). Recall that in the CAPM the expected returns of an asset are given by:

$$ E[R_a] = R_f + \beta_a (E[R_m] - R_f) $$

where $R_f$ is the risk free rate, $E[R_m]$ is the expected market return, and $\beta_a$ is the asset beta ($\beta_a = \text{cov}(R_a, R_m) / \text{var}(R_m)$). Thus, the model presented in this section suggests that the market is a strong determinant of individual asset returns. Furthermore the market or common factor possesses leverage, which could be interpreted as a ‘flight to quality’: when the market shows negative returns, agents eventually choose to invest their assets in other instruments than stocks. This highly correlated move rises the volatility in the stock market. In case of an upturn however, the investors react slower
3. Gain/Loss Asymmetry and the Leverage Effect

and reinvest in individual assets in a less synchronized manner, leading to a smaller increase in overall volatility. The following section shows what happens, when one introduces some leverage at the individual stock level.

In indices the properties of the market factor are amplified due to their synchronization across assets. As it was shown with this model, the properties of the indices converges towards those of the market with increasing cross-correlation. The convergence is faster for the index than for the individual asset, as shown in figure 3.9. This acts in some way similarly to filtering: the individual components of the assets are diluted away as the returns are aggregated in an index. As the correlation tends towards one, the market and the assets can’t be distinguished. As described above, because individual stock returns are based on the market returns, they also display some leverage and gain/loss asymmetry at the asset level. This duality is interesting, as stocks depend on the market that they compose. The latter possessing to some extent an emerging property in gain/loss asymmetry, due to asset cross-correlations. Logically, increasing the number of assets also leads to a stronger convergence towards the market properties, decreasing the impact of individual innovations.

Finally, the model shows that we can reproduce most of the empirical results with a single mechanism. Introducing leverage at the market level, on which individual asset returns are dependent is sufficient to reproduce the amplification of gain/loss asymmetry and the leverage effect in the index as compared to in the individual stock. However, the contemporaneous correlation between returns and volatility observed at the individual level is not reproduced in this model. As suggested in the previous section, this feature is not due to the leverage effect and is based on a different mechanism.

3.2.2 The Inverse Model

An inverse model for stock correlation can be built to assess the collective origin of the leverage effect. We have found that with increasing correlation the between its constituents, the index reaches the characteristics of the underlying EGARCH process $X(t)$. This can be seen as a common market factor that has a strong leverage effect. But, as noted in the previous chapter, stocks do possess some asymmetry and the leverage effect. In this subsection, we build an inverse model, in which the underlying market factor $X(t)$ does not possess any leverage. The stocks do however, and react to their performance against the market: if the stock underperforms the market, its volatility will increase. If we let the underlying factor $X(t)$ be a geometrical
Brownian motion with standard deviation $\sigma_x$:

$$X(t) = \xi(t), \quad i = 1, ..., N \quad (3.12)$$

$$Y_i(t) = \alpha X(t) + \epsilon_i(t) \quad (3.13)$$

$$\log(\sigma_{i,t}^2) = a_0 + a_1 \frac{\epsilon_{i,t-1}}{\sigma_{i,t-1}} + a_1 b \left( \frac{\epsilon_{i,t-1}}{\sigma_{i,t-1}} - E \left[ \frac{\epsilon_{i,t-1}}{\sigma_{i,t-1}} \right] \right) + b_1 \log \sigma_{i,t-1}^2 \quad (3.14)$$

where $\xi(t) \sim \mathcal{N}(0, \sigma_x^2)$ are i.i.d random variables with standard deviation $\sigma_x$, and $\epsilon_i(t), i = 1, ..., N$ are i.i.d. innovations $\epsilon_i \sim \mathcal{N}(0, \sigma_i(t)^2)$. The assets $Y_i$ are univariate EGARCH processes with expectation $\alpha X(t)$. The volatility $\sigma_i(t)$ depends only on the innovations $\eta_i$. This means that the volatility increases if the asset does not beat the market, even if the overall return $Y_i(t) = X(t) + \epsilon_i(t)$ is positive.

As previously, the parameter $\alpha$ allows to control the correlation between the assets $Y_i$. The correlation between $Y_i$ and $Y_j$ is approximated by (derivation in Appendix D):

$$\text{corr}(Y_i, Y_j) = \frac{\alpha^2 \sigma_x^2}{\sqrt{(\alpha^2 \sigma_x^2 + \sigma_i^2)(\alpha^2 \sigma_x^2 + \sigma_j^2)}} \quad (3.15)$$

setting the EGARCH(1,1) parameters such that the unconditional volatilities of the assets are equal one to another ($\sigma_i^2 = \sigma_j^2$):

$$\text{corr}(Y_i, Y_j) = \frac{\alpha^2 \sigma_x^2}{\alpha^2 \sigma_x^2 + \sigma_i^2} = \frac{\alpha^2}{\alpha^2 + \sigma_i^2 / \sigma_x^2} \quad (3.16)$$

if we also choose to set the variance of $X(t)$ to $\sigma_x^2 = \sigma_i^2$, we obtain the same form as before:

$$\text{corr}(Y_i, Y_j) = \frac{\alpha^2}{\alpha^2 + 1} \quad (3.17)$$

Logically, asymmetry and leverage disappear with increasing asset correlation and converges to the properties of the common market factor. Similarly to the ‘normal’ model, with increasing correlation between assets, gain/loss asymmetry displayed converges to the properties of $X(t)$ (see figure 3.12). Starting at $\tau_+^* = 15$ and $\tau_-^* = 9$, for gains and losses respectively, both converge to $\tau_+^* = 10$. The leverage effect decays from an initial maximum of $\max L(\tau) = 0.15$ to 0. This initial value of the maximum leverage in the index, with zero asset correlation, is already lower than the individual values of around 0.4. The stocks converge to the common market factor, which is a geometrical Brownian motion, and therefore do not display any leverage or asymmetry. The correlation parameter $\alpha$ is large enough.
to increase the relative importance of $X(t)$ over the individual innovations. Furthermore, the innovations are i.i.d, making the leverage effect displayed by any stock is independent from the others. Therefore, increasing inter asset correlation does not lead to an increase of the leverage effect but to the inverse result: individual leverage effects compensate each other, leading to their dilution in the index and ultimately disappearance.

Conclusion

The results showed in the inverse model are rather obvious. As cross-correlation between assets is increased, the common market factor $X(t)$ gains in importance. Because this common market factor is a pure geometrical Brownian motion, there is no mechanism whatsoever able to reproduce leverage or gain/loss asymmetry. These properties, which are present at the individual level, are rapidly erased when measured as part of an index. This confirms that there is a strong convergence towards to properties of the common market factor, as shown in the previous market model.

3.2.3 Dynamics of Convergence

More constituents increase the sensitivity of the index to changing intercorrelations. Increasing the number of assets increases the convergence of the index properties to those of the underlying, but how does this impact convergence with increasing correlation? Looking at it from a investor perspective, the goal of a diversified portfolio is to decrease average cross-correlation in order to decrease unsystematic risk. This is done at the cost of more assets in the portfolio. Recall that the standard deviation of a portfolio,
3.3. Impact of the Leverage Effect

A common measure of its riskiness is given by:

\[ \sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{ij} \]  

(3.18)

where \( \sigma_p^2 \) is the portfolio variance, \( \sigma_i^2 \) and \( w_i \) are the individual asset variances and weights respectively, \( \rho_{ij} \) is the cross-correlation between assets \( i \) and \( j \). If the cross-correlation \( \rho_{ij} \) is low, it reduces the portfolio variance. However, the results of the previous subsections show that the inverse statistics and the leverage effect of the common factor \( X(t) \) behave more like systematic risk, the risk associated with aggregate market returns. Therefore, it cannot be diversified away in a portfolio. Increasing the number of assets would increase the convergence of the portfolio returns to that of the underlying factor \( X(t) \). Figure 3.13 shows that this is indeed the case in this model. The left column shows the standard correlation model introduced above, while the right column shows the results of the inverse model. The first four plots are the results of a 2 asset portfolio, the following four are simulations of a 100 asset portfolio. Convergence towards the properties of the underlying common factor \( X(t) \) is nearly linear in the 2 asset portfolio (these should be contrasted to figures 3.10 and 3.12). On the other hand the convergence with increasing correlation is much faster in the 100 asset portfolio (figure 3.13, lower four plots). This confirms the importance of the number of constituents in a portfolio when studying its leverage effect and inverse statistics.

Conclusion

The results of this subsection are also easily predicted: the more assets are present in the portfolio or index, the faster the convergence towards the common factor \( X(t) \). Individual innovations are diluted in the index, as their contribution is weighed as \( \frac{1}{N} \). This leads to some ‘filtering’ in which the common features are conserved during the averaging. As \( N \) is increased the filtering is more powerful and the noise present at the individual level is gradually erased. The border values are reached in all cases, but the dynamics are different. In a 100 asset portfolio, very little asset cross-correlation is necessary to fully recover the properties of the underlying market factor. This leads to an interesting observation: diversification by the addition of assets actually exposes an investor much stronger to the underlying market factor as if he were to hold fewer of such assets.

3.3 Impact of the Leverage Effect

In this section we are interested in assessing the impact of the leverage effect on gain/loss asymmetry. Independently from the differences between
3. Gain/Loss Asymmetry and the Leverage Effect

Figure 3.13: Left column are the results of the ‘normal’ correlation model, the right column those of the inverse model. (top four plots) 2 asset portfolio. The convergence towards the properties of the underlying common factor $X(t)$ is nearly linear for the optimal investment horizons and for the maximum of the observed leverage. (lower four plots) 100 asset portfolio. If there are many more constituents in the index the convergence is much faster and shows a sigmoid behavior. The feature is particularly striking for the leverage effect (lower 2 plots).
3.3. Impact of the Leverage Effect

stocks and indices, some other properties of the leverage effect could explain why we do not always observe the same gain/loss asymmetry although the amplitude of the leverage effect is rather similar.

The section focuses on the time decay of the leverage effect and the contemporaneous leverage effect. The first relates to the regime theory introduced by Ahlgren et al. [56]: different regimes may display different behaviors in the time decay of the leverage effect. As discussed in section 3.1.2, the contemporaneous correlation between returns and volatility is not the same as mechanism as the leverage effect.

3.3.1 Time Decay of the Leverage Effect

The time decay of the leverage effect is an additional explanation to its amplitude in explaining the gain/loss asymmetry observed. As introduced in subsection 3.1.3 on page 38, the maximum amplitude of the leverage effect does not seem to explain entirely the gain/loss asymmetry observed in the stocks or indices. We hinted at time decay of the leverage function as a possible explanation for the inverse statistics seen in the empirical results. In this subsection we investigate the impact of volatility autocorrelation on the time decay of the leverage function and its relation to gain/loss asymmetry. Using the correlation model introduced in the beginning of this section, we vary the parameter $b_1$, which is responsible for the autocorrelation of past volatility. Recall that the equation for conditional volatility in the EGARCH(1,1) specification used is:

$$
\log(\sigma_t^2) = a_0 + a_{1a} \frac{\varepsilon_{t-1}}{\sigma_{t-1}} + a_{1b} \left( \frac{\varepsilon_{t-1}}{\sigma_{t-1}} - E \left[ \frac{\varepsilon_{t-1}}{\sigma_{t-1}} \right] \right) + b_1 \log(\sigma_{t-1}^2) \quad (3.19)
$$

However, as can be seen in Appendix D on page 78, increasing $b_1$, while holding all the other parameters constant, leads to a rise in the maximum leverage effect observed. To isolate the effect of volatility autocorrelation from that of the amplitude of the leverage effect, we tune leverage parameter $a_{1a}$ to keep the maximum of the leverage function constant. Using a 10 asset index the leverage function for $b_1 = 0.8 \rightarrow 0.95$ (red to purple curves) is depicted in figure 3.14.

A slower decay in the leverage function leads to a spread in the optimal investment horizons. The left plot of figure 3.15 shows the increasing gap between the optimal investment horizons for gains and losses with slower decay in the leverage function. As the volatility autocorrelation parameter $b_1$ is increased from 0.8 to 0.95, $\tau_{+}^* + \rho$ is driven away from 18 to 28 days. Interestingly, the optimal investment horizon for losses remains more or less constant at $\tau_{-}^* \sim 8$ days. The right plot shows that the maximum leverage is held constant by varying the EGARCH(1,1) parameter $a_{1a}$.
3. Gain/Loss Asymmetry and the Leverage Effect

Figure 3.14: Leverage effect for a 10 asset index with the volatility autocorrelation parameter varying from $b_1 = 0.8 \rightarrow 0.95$. The leverage parameter $a_{1a}$ was tuned to keep the maximum leverage effect constant.

Figure 3.15: Spread between optimal investment horizons increases as the autocorrelation between present and past volatility is increased. The time decay of the leverage function becomes slower and leads to an increase in gain/loss asymmetry. The maximum leverage is held constant by tuning the EGARCH leverage parameter $A_{1a}$.

A slower decay in the leverage function affects the distribution of waiting times to reach $+\rho$ but has a much smaller impact on the one for the lower barrier $-\rho$. Figure 3.16 shows the distribution of waiting times for $b_1 = 0.8 \rightarrow 0.95$ (red to purple curves). It can be seen in the left plot that the inverse statistics for gains is strongly affected by the change in volatility autocorrelation. As noted above, the optimal investment horizons for gains drifts away from $\tau \simeq 18$ days to $\tau \simeq 28$ days. Furthermore, the probability distribution becomes skewed. The frequency of investment horizons faster than $\sim 9$ days increases whereas those larger decrease. Interestingly, the value of 9 days corresponds to the average optimal investment horizon $\tau_{-\rho}$ for losses. The changes in the distribution of waiting times for losses (right
plot) are much more subtle. The optimal investment horizon hardly moves at all, and the probability distribution does not face any major changes in shape.

Figure 3.16: Inverse statistics for gains (left) and (losses) at a $\rho = 5\sigma$ return level. Red to purple curves correspond to $b_1 = 0.8 \rightarrow 0.95$. The distribution of waiting times for gains is much more affected by the change in the time decay of the leverage function than the distribution for losses.

Conclusion

The decay of the leverage function introduces a alternative explanation to gain/loss asymmetry than simply its amplitude. The fact that a slower decay in the correlation between past volatility and present returns leads to an increase in gain/loss asymmetry can be explained as follows: negative returns make the price level come closer to the lower barrier $-\rho$ and lead to an increase of volatility, which in turn increases the probability of crossing that barrier. A slower decay in the leverage function increases the impact of past returns. Thus the increase in volatility raised by negative returns is more persistent and leads to a further increase of the probability of crossing $-\rho$. Similarly, positive returns decrease volatility making it harder to cross the level $+\rho$ in the later future.

Different time periods or different markets may possess similar amplitude of the leverage effect but display very different patterns in gain/loss asymmetry. The time decay of the leverage effect offers an alternative explanation to explain why markets may react differently one from another or why other instruments such as bonds or even housing prices may display different gain/loss asymmetry patterns. Similarly, different regimes may display different decays and thus show different behaviors.
3.3.2 Contemporaneous Leverage Effect

The difference in the onset of the leverage function between individual stocks and indices may denote a different underlying mechanism. As noted in subsection 3.1.2 on page 32, the maximum of the leverage effect is located at $\tau = 0$ for individual stocks, whereas for indices it is at $\tau = 1$. Whereas the latter can be explained by the leverage effect, and is reproduced nicely by the model presented in this section, it is not possible to obtain an onset at $\tau = 0$ with the same specification. Going back to the definition of the leverage effect, a value of $L(\tau = 0) < 0$ implies negative returns are correlated with higher volatility. Such a contemporaneous correlation can be obtained in two different manners:

- The distribution of returns is biased towards negative values.
- The distribution of returns is negatively skewed.

Note however that when we built an equally weighted index of 10 stocks, the effect is still present and even amplified (see subsection 3.1.3 and figure 3.5). This might suggest supplementary properties of real indices that were not built in our artificial model. Real indices often have different weighting schemes and their composition is rebalanced, for example. The effect of a bias and a skew in individual returns and on the leverage effect and on gain/loss asymmetry is assessed in the following paragraphs.

### Bias in Innovations

We assess the relation between the contemporaneous leverage effect and a bias in individual innovations with the following modified common factor model. We build the index log returns $I(t) = \frac{1}{N} \sum_{i=1}^{N} Y_i(t)$ as in section 3.2, with the addition of a bias in individual innovations $b$, which are measured as a fraction of $\sigma$:

$$Y_i = \alpha X + \xi_i + b, \quad \xi_i \sim N(0, \sigma^2), \quad i = 1, ..., N \tag{3.20}$$

$$X(t) = \epsilon(t), \quad \epsilon(t) \sim N(0, \sigma_{x,t}^2) \tag{3.21}$$

$$\log(\sigma_{x,t}^2) = a_0 + a_{1a} \frac{\epsilon_{t-1}}{\sigma_{x,t-1}} + a_{1b} \left( \frac{[\epsilon_{t-1}]}{\sigma_{x,t-1}} - E \left[ \frac{[\epsilon_{t-1}]}{\sigma_{x,t-1}} \right] \right) + b_1 \log \sigma_{x,t-1}^2 \tag{3.22}$$

where $\alpha$ is the correlation parameter to the common factor $X(t)$.

A bias in the individual innovations is linearly correlated with the value of $L(\tau = 0)$. Figure 3.17 shows the relation between $L(\tau = 0)$ and $b$. The red squares represent a 20 asset portfolio with $\alpha = 0.5$, whereas the blue circles represent the average leverage effect of these 20 assets. As expected, we observe a linear correlation between the drift parameter and the leverage
3.3. Impact of the Leverage Effect

Figure 3.17: Linear correlation between the bias in the individual innovations and the value of the leverage effect for $\tau = 0$. The slope of the linear fit is stronger for the index built with the 20 stocks (red) than for the average of the individual leverage effects (blue). The slopes of the linear fits are 3.54 and 1.745 for the index and stocks respectively.

The effect for $\tau = 0$. The effect is amplified in the index, where the slope is twice as strong (see caption of figure 3.17). The slope and the effect does not depend on the number of assets in the portfolio (data not shown).

Figure 3.18: Optimal investment horizons for stocks (left) and the artificial index (right). Both show a maximum for intermediate values. The distribution is asymmetric and peaked for the index, where the effect of the drift is amplified. The maximum for the distribution of $\tau_{\rho}$ is shifted to the left.

The optimal investment horizons are affected by the bias in the individual innovations. We saw that the leverage effect for $\tau = 0$ is linearly dependent on the drift parameter, but how does it affect gain/loss asymmetry? Figure 3.18 shows the results for the individual stocks (left) and the 20 asset index (right). We can see in both cases that the distribution of the optimal investment horizons in function of the drift has a maximum for intermediate values of bias. Furthermore, while the distribution for gains
has its maximum close to \( b = 0 \), the distribution for losses is shifted to negative values of \( b \). As it could be predicted from the previous figure, the effect is widely amplified in the index, where the distribution has a much more pronounced maximum. The maxima for the index are located at \( b = -0.15 \) and \( b = 0 \) for losses and gains respectively. For stocks the curve does not possess a precise maxima. Recall that because we are studying closing prices, the optimal investment horizons can only take discrete values. Another striking feature is the apparent asymmetry in the distribution of \( \tau^+_{\rho} \); the left side is very steep and drops nearly instantaneously to a value of 2 days, whereas the right side decays more progressively to converge at 6 days.

![Figure 3.19](image)

**Figure 3.19:** Asymmetry ratio for stocks (blue) and the index (red). The asymmetry ratio for the stocks is close to zero, but shows a slight upward trend. For the index, the dependence of the asymmetry ratio on the drift is nonlinear. After a maximum of 1.5, situated at \( b = 0 \), it decay progressively to \( r_{\rho} = 0.5 \). For negative drifts, the asymmetry ratio is even negative and converges at \( r_{\rho} = -0.7 \).

The gain/loss asymmetry can even be reversed if the drift is sufficiently negative. As seen in figure 3.19, the asymmetry ratio for stocks has a slight positive linear trend: a positive bias in the innovations seem to trigger some asymmetry in the stocks. The effect is strongly nonlinear for the index. The maximum of \( r_{\rho} = 1.5 \) is found for \( b = 0 \) and decays progressively until \( r_{\rho} = 0.5 \) or \( \tau^+_{\rho} = 1.5 \tau^\rho \). For \( b < 0 \), the asymmetry declines sharply and is negative for \( b < -0.075 \). The asymmetry ratio for these values then converges at \( r_{\rho} \simeq -0.7 \). Again, the effect of a bias in the individual innovations is much stronger for the index than for the individual stocks.

**Skewness**

To assess the impact of skewness in returns on inverse statistics we run simulations with the model described above, without bias and with
3.3. Impact of the Leverage Effect

Innovations following a lognormal distribution:

\[ Y_i = \alpha X + \xi_i + \Delta, \quad \xi_i \sim \mathcal{N}(0, \sigma^2), \quad i = 1, \ldots, N \]  

(3.23)

where \( \Delta = \exp((\ln \mu + \sigma^2/2)) \) is used to set the expectation of the returns to zero: \( \mathbb{E}[Y] = 0 \). To assess negative skewness we use the symmetry by the vertical axis of the given distribution.

Skewness in the individual returns also influences the amplitude of the leverage effect for \( \tau = 0 \). Using the specification introduced here above we obtain an interesting impact on the leverage effect for \( \tau = 0 \), as shown by figure 3.20. The left figure shows the usage of the usual lognormal (positive skew), whereas the right side depicts the mirrored lognormal (negative skew). In the index (red squares) the effect is diluted and the dependence is not noticeable, except for extreme values of skew, for which there is a positive correlation. For the individual stocks (blue circles) on the other hand, the dependence is noticeable for \( \sigma_{\ln N} > 0.3 \), both in cases of positive skew (left plot) and with the mirrored lognormal distribution (right plot, negative skew). In these two cases there is a positive correlation between the skewness and the position of \( L(\tau = 0) \). Note also the amplitude of the leverage effect for \( \tau = 0 \) is quite large for very skewed distributions (the vertical scale on the figures goes from -3 to 3). Because the empirical results showed that the leverage effect for \( \tau = 0 \) is usually negative, we will focus in the remaining of this section on the negative skew.

Negative skewness also impacts the position of the optimal investment horizons. Repeating the procedure while recording the position of the
3. Gain/Loss Asymmetry and the Leverage Effect

optimal investment horizons leads to the plots in figure 3.21. The simulations are run with the mirror of the lognormal (negative skewness), because we observed negative values of \( L(\tau = 0) \) in the empirical section of this work. The most striking feature is the difference between individual stocks (left plot) and the index (right plot). The position of \( \tau^*_{-\rho} \) does not vary much, whereas the position of \( \tau^*_{+\rho} \) shows very different behavior when assessed individually or as part of an index. When assessed in stocks, the optimal investment horizon for gains decreases from 22 to 14 days with increasing negative skewness. However, for values of \( \sigma_{lnN} > 0.7 \) the optimal investment horizon for gains increases back up to 16 days. In the index the effect is monotonous. \( \tau^*_{+\rho} \) decreases from 25 to 17 days with increasing skewness.

Asymmetry ratio is strongly dependent on the skewness. From the simulations described above, one can plot the asymmetry ratios for stocks (blue circles) and the index (red squares) to compare their behavior (see figure 3.22. As could be deducted from the optimal investment horizons alone, the asymmetry ratio in the index is decreasing with increasing negative skewness. For stocks, there is a minimum situated between \( 0.475 < \sigma_{lnN} < 0.7 \). The maximum of gain/loss asymmetry is obtained in both cases when the skew is absent.

Conclusion

A bias in the individual innovations does not seem to fit correctly the empirical observations. While a bias in returns does impact the value of \( L(\tau = 0) \), the fact it is amplified at the index level does not match the
3.3. Impact of the Leverage Effect

Figure 3.22: The asymmetry ratio for stocks (blue circles) and indices (red squares) is strongly dependent on the skew in the individual innovations. The horizontal axis represents the standard deviation of the lognormal distribution used to simulated the innovations. One can see that the effect has a faster impact at the individual level, for which the curve is convex. The index, on the other hand, shows a monotonous concave decrease. Both curves converge towards an asymmetry ratio of approximately 0.8.

observations made in the previous sections of this work. Recall that we observed the effect disappears when it is measured at the index or portfolio level. A bias would however make sense in a purely macroeconomic point of view, in which there is a general price inflation and overall growth. Recall that in the empirical section we do not detrend the financial time series, which tend to grow exponentially.

Skewness in the individual returns follows well what is observed empirically. Adding skewness to the individual returns shows similar characteristics at the stock level but in addition vanishes at the index level. A negative skewness, which corresponds to the values of \( L(\tau = 0) < 0 \) shown in the section 3.1.2, means gains are more frequent but have a smaller amplitude than the large less frequent losses. Considering we are observing adjusted closing prices, we do not capture the intraday trading dynamics and measure only its consequence as a difference between the opening and the closing prices. Perhaps negative intraday returns lead to some herding or panic effect which creates larger losses by some feedback mechanism. On the other hand, gains do not lead to such effects and display smaller amplitudes on average.

In this section we assessed two different mechanisms that lead to a contemporaneous correlation between volatility and stock returns observed empirically. This section shows that the mechanism is most probably due to some skewness in the individual stock returns if there are considered
3. GAIN/LOSS ASYMMETRY AND THE LEVERAGE EFFECT

independently. However, due to the observations that the financial time series display some positive trend, we might have in reality a mixture of both of these mechanisms. Furthermore, a slight positive trend may be large enough to mask some positive skew in the innovations. To distinguish between these two cases, further simultaneous studies of the leverage effect and the distribution of the returns could be considered.
Chapter 4

Conclusion

This paper investigated in three different steps the relations between gain/loss asymmetry and the leverage effect and their amplification in equity indices. First, using only past financial time series, empirical facts concerning inverse statistics and return-volatility correlation are established. Then a model of stock correlation shows how asset synchronization leads to an amplification of these properties in an equity index or a portfolio. Finally, using synthetic time series, we study the impact of variations in the leverage effect function on gain/loss asymmetry at the individual asset level.

4.1 Conclusions

Empirics

More work must be conducted in finding a functional form to fit the empirical probability distributions of the waiting times. The generalized inverse Gamma function is extremely flexible and was suggested by Simonsen et al. to fit the empirical distributions of waiting times [38]. In this work we find it extremely difficult to find appropriate parameters to validate such fits. More work is required to question its validity and/or bring another more adapted functional form.

Stocks and equity indices display distinct properties and behave differently with regard to gain/loss asymmetry and the leverage function. Over a given period, the indices show stronger gain/loss asymmetry and have a larger maximum in their leverage function. This seems to be in agreement with Siven and Lins [54], who showed with an EGARCH model that a parameter responsible for asymmetric responses to returns - a form of leverage effect - leads to larger gain/loss asymmetry. Therefore, larger gain/loss asymmetry in indices may be linked to the larger leverage effect, which is measured in those. We suggest that stock correlation leads to a
**4. Conclusion**

A stronger leverage effect in an index than in individual stocks and leads to larger gain/loss asymmetry.

An artificial equity index made of S&P constituents gives more insight on the relations between the inverse statistics of the index compared to those of its constituents. This study confirmed what we have seen in the preceding section. However, because we are now comparing the index directly with its own constituents, we eliminate some uncertainties that were left. The results are similar: gain/loss asymmetry and the leverage effect are amplified in the index compared to the average of the constituents. Stock cross-correlation between the constituents may be responsible for this amplification of the leverage effect at the index level. However, we question the differences in amplification between gain/loss asymmetry and the leverage effect however using these artificial indices. Because the difference in gain/loss asymmetry is smaller than what we expected from the large difference in the leverage effect, perhaps additional mechanisms modulating its amplitude come into play. We suggest that a difference in the time decay of the leverage effect in stocks and indices could be an appropriate explanation: stocks have less *memory* than markets in the leverage function, it displays less volatility autocorrelation.

**Stock Correlation Model**

The market model introduced in section 3.2 shows that the synchronization of assets leads to a amplification of the leverage effect and gain/loss asymmetry in an equity index. When increasing cross-correlation of assets, which are based on a common market factor displaying leverage, the leverage effect and gain/loss asymmetry is amplified at the index level. In fact, the properties of the individual assets and the index they compose converge to those of the underlying market factor. The subtlety comes from the fact the convergence is much faster at the index level. Such a specification of the expected returns on the assets reminds the one of the capital asset pricing model:

\[
E[R_a] = R_f + \beta_a (E[R_m] - R_f)
\]

where \(R_f\) is the risk free rate, \(E[R_m]\) is the expected market return, and \(\beta_a\) is the asset beta (\(\beta_a = \text{cov}(R_a, R_m) / \text{var}(R_m)\)). The leverage in the underlying *market factor* suggests a mechanism of ‘flight to quality’: when the market shows negative returns, agents eventually choose to invest their assets in other instruments than stocks. This highly correlated move rises the volatility in the stock market. In case of an upturn however, the investors react slower and reinvest in individual assets in a less synchronized manner, leading to a smaller increase in overall volatility. This effect is present on the individual scale, but because the individual innovations are filtered out, it is
4.1. Conclusions

amplified when measured in an index. Specifying a leverage mechanism at
the individual levels and setting the market factor to a geometrical Brownian
motion does not lead to an amplification of the properties, but to their
disappearance. This does not however, mean that there is no leverage effect
at the individual level on risky stocks in addition to the one observed in the
market factor.

Logically, the dynamics of convergence towards the market factor are
further amplified if the index has a large number of constituents. The
goal of a diversified portfolio is to decrease average cross-correlation
in order to decrease unsystematic risk, which is the company-specific
or industry-specific risk in a portfolio. This is usually accomplished
at the cost of including more assets in the portfolio, if possible with
low cross-correlation one with another. Doing so dilutes the individual
innovations away and therefore decreases unsystematic risk. In the same
time, this leads to a convergence towards the marker properties, in this
case leverage. As described previously, higher leverage is correlated with
stronger gain/loss asymmetry. The cost of diversification is convergence
towards the underlying market factor, which displays a strong leverage
effect. An investor may be increasing his waiting times for gains compared
to those for losses, in exchange for obtaining lower portfolio risk.

The Leverage Effect

A slower decay of the volatility autocorrelation in the leverage effect leads
to stronger gain/loss asymmetry. The amplitude of the leverage effect
is closely related to gain/loss asymmetry, however the time decay of the
volatility autocorrelation modulates its influence. Using synthetic time
series, we showed that increasing the autocorrelation of the volatility, all
other factors held constant, leads to an increase in gain/loss asymmetry.
Stronger autocorrelation implies a slower decay back to the unconditional
variance of the synthetic asset price process. Increases in volatility due
to negative shocks last longer and increase the probability of crossing the
return levels $\pm \rho$. However, because a negative return brings the asset price
closer to the bottom level $-\rho$, it has a higher probability of being crossed
than $+\rho$. This would amplify the leverage effect and in turn gain/loss
asymmetry. Furthermore, a positive return would decrease the conditional
variance and reduce the impact of the increased volatility. In asset
price processes, alternate regimes may display different autocorrelations in
volatility and may lead to different inverse statistics although the amplitude
of the leverage effect remains similar.

The contemporaneous correlation between returns and volatility seen in
individual assets suggests individual asset returns are negatively skewed.
Two mechanisms are possible to give rise to a negative correlation between
returns and volatility at the individual level: a negative bias in the returns or some skewness in their distribution. In both cases this would mean large returns are often negative. However, only a skewness in the individual returns disappears at the index level, as observed empirically. The data assessed in this work is adjusted closing prices, before which many trades have been made. This means the negative skewness at the individual level may be explained by some intraday herding behavior in the case of negative returns. Investors often place ‘stop-loss’ orders, which add some selling pressure on a falling asset price, and would lead to larger, less frequent losses. The gains would on the other hand be more frequent because of the growing economy, but smaller in amplitude due to the absence of intraday herding. Overall the expectation of the returns may be positive or negative, the effect thereof would be seen at the index level.

4.2 Summary of Contributions

This work showed empirically that there is a correlation between the amplitude of gain/loss asymmetry and of the leverage effect. It also confirms that the amplitude of both is larger in indices than in individual stocks, and suggests that it is the cross-correlation of assets within the portfolio or index that leads to this amplification. This hypothesis is tested and verified in a common market factor model, for which the dynamics are studied. Finally, this work shows how variations of the leverage effect function can recreate gain/loss asymmetry patterns that are observed empirically.
Chapter 5

Future Perspectives

5.1 Functional Form of Inverse Statistics

As described here above, more work must be conducted on finding an appropriate functional form to fit the empirical distributions of waiting times. Such functional forms would be helpful in identifying the underlying mechanisms leading to these modified first passage time problems.

5.2 Diamond Leverage Effect

![Figure 5.1: Diamond-like behavior of the leverage effect function for 9 stocks of the S&P 500. The function is positive for $\tau = -6, -1, 4$ and negative for $\tau = -4, 1, 6$. The amplitude of the function is also five times superior to what was observed in the average of the other stocks.](image)

Although most of the stocks display a leverage effect qualitatively similar to the average presented in this work, some of them have a common, radically different behavior. While screening through the stocks of the S&P
5. Future Perspectives

500 it appears that nine of them have a common and distinct behavior (see figure 5.1 and table 5.1). For all nine stocks listed the leverage effect function is zero except for $\tau = -6, -1, 4$, for which it is positive and $\tau = -4, 1, 6$, for which it is negative. A non zero value for $\tau < 0$ implies past volatility and future returns are correlated. If it is so there would be some possibility to predict future returns based on past volatility. In addition, the symmetry and the reproduction of the exact same pattern over nine different stocks is interesting and is worth to investigate.

<table>
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<td>Healthcare</td>
</tr>
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<td>EFX</td>
<td>Equifax Inc.</td>
<td>Services</td>
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<td>Healthcare</td>
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<td>PKI</td>
<td>PerkinElmer, Inc</td>
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<tr>
<td>STZ</td>
<td>Constellation Brands, Inc.</td>
<td>Alcoholic Beverages</td>
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</table>

Table 5.1: Companies of the S&P 500, which display a diamond-like leverage effect function.

5.3 Asset Beta and Gain/Loss Asymmetry

The asset beta could be an additional parameter against which gain/loss asymmetry and the leverage effect can be assessed. In the market model introduced in this paper, we show that the more an asset is correlated to the market factor, the closer their properties are. The asset beta is a measure of correlation between the market and a given asset. Therefore it would be interesting to study gain/loss asymmetry and its relation to the unleveled asset beta.

5.4 Regime Transitions

A model to detect regime transitions based on the amplitude of the leverage effect and of the gain/loss asymmetry could be investigated. At the end of section 2.2.2, we presented the regime theory suggested by Ahlgren et al [56]. They introduce the Gaussian Mixture model that shows that a simple autocorrelated structure in state occurrences is sufficient to generate gain/loss asymmetry with processes that do not possess these characteristics. If we suppose that a certain gain/loss asymmetry ratio relates to a given leverage effect amplitude, we can detect whether the
5.4. Regime Transitions

gain/loss asymmetry observed is above or below the level predicted. By doing so one could detect ‘anomalies’ in gain/loss asymmetry, which can be assimilated to regime transitions. Such a model would be very valuable in the area of modeling and prediction of financial time series.
Chapter 6

Acknowledgements

I would like to thank Dr. Vladimir Filimonov and Prof. Didier Sornette for their help and guidance during the completion of my master’s thesis. First of all I would like to thank Vladimir Filimonov for his supervision. His attention for detail, coupled with great analytical skills brought me extensive feedback for the completion of my work. I also thank Prof. Didier Sornette for the opportunity to do my master thesis at his chair. His ideas and the conviction with which they are expressed caught my attention during classes. Working in his department was a great opportunity to get closer to the studies accomplished in by himself and his researchers.

I also thank all my fellow students for their input and advice during these six months, especially Peter Zeeuw van der Laan and Zalán Forró. Best wishes to both of them for their career start and PhD respectively.
Appendix A

S&P500 Constituents

The four indices used in section 3.1.2 are listed in table A.1

<table>
<thead>
<tr>
<th>Description</th>
<th>Ticker</th>
<th>Stocks</th>
<th>Weight Type</th>
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<td>DJIA</td>
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<td>price</td>
</tr>
<tr>
<td>Standard &amp; Poors 500</td>
<td>S&amp;P500</td>
<td>500</td>
<td>free-float capitalization</td>
</tr>
<tr>
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<td>NDX</td>
<td>100</td>
<td>capitalization</td>
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<tr>
<td>Russell 1000</td>
<td>RUI</td>
<td>1000</td>
<td>free-float capitalization</td>
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</tbody>
</table>

Table A.1: Indices used in section 3.1.2 and in figure 3.1

The stocks used to compute the inverse statistics shown in figure 3.1 are listed on page 70.
### S&P500 Constituents

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<td>LSI</td>
<td>NU</td>
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**Table A.2:** Stocks used in section 3.1.2 and in figure 3.1
Appendix B

Leverage Function Form

The results shown in the table here under were obtained by fitting by the least squares method (MATLAB: polyfit) to the logarithm of the leverage effect for $\tau > 0$ versus $\tau$ and $\ln \tau$ for the exponential and power law fit, respectively. Due to the poor quality of the statistics it is difficult to conclude without ambiguity on the best fit for the form of the leverage function.

<table>
<thead>
<tr>
<th>$L(\tau)$, $\tau &gt; 0$</th>
<th>Stocks</th>
<th>Indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential fit</td>
<td>$-0.12 \exp\left(-\frac{\tau}{40.98}\right)$</td>
<td>$-0.22 \exp\left(-\frac{\tau}{30.77}\right)$</td>
</tr>
<tr>
<td>Power law fit</td>
<td>$-0.2297 \tau^{-0.427}$</td>
<td>$-0.4438 \tau^{-0.512}$</td>
</tr>
</tbody>
</table>

Figure B.1: Exponential and power law fitting on leverage function $L(\tau)$, as shown in subsection 3.1.2.
Appendix C

Artificial Index

The following stocks were used in subsection 3.1.3 for the creation of artificial indices.

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<th>Name</th>
<th>Description</th>
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<tbody>
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<td>AA</td>
<td>Alcoa Inc.</td>
<td>Aluminum producer</td>
</tr>
<tr>
<td>BA</td>
<td>The Boeing Company</td>
<td>Aircraft manufacturer</td>
</tr>
<tr>
<td>C</td>
<td>Citigroup Inc.</td>
<td>Multinational financial services corporation</td>
</tr>
<tr>
<td>DD</td>
<td>DuPont de Nemours and Company</td>
<td>Chemical Company</td>
</tr>
<tr>
<td>GE</td>
<td>General Electric Company</td>
<td>Energy, Technology Infrastructure, Capital Finance and Consumer &amp; Industrial Technology and consulting</td>
</tr>
<tr>
<td>IBM</td>
<td>International Business Machines Corporation</td>
<td></td>
</tr>
<tr>
<td>JNJ</td>
<td>Johnson &amp; Johnson</td>
<td>Pharmaceutical, medical devices and consumer packaged goods</td>
</tr>
<tr>
<td>KO</td>
<td>The Coca-Cola Company</td>
<td>Beverage manufacturer</td>
</tr>
<tr>
<td>MCD</td>
<td>McDonald’s Corporation</td>
<td>Restaurants</td>
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<tr>
<td>PG</td>
<td>Procter &amp; Gamble</td>
<td>Consumer goods</td>
</tr>
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</table>

*Table C.1: Companies used in creation of imaginary indices from empirical time series*
Appendix D

Correlation Model

D.1 Specification

The model runs simulations of log returns $I(t)$ specified as:

$$I(t) = \frac{1}{N} \sum_{i=1}^{N} Y_i(t)$$  \hspace{1cm} (D.1)

where $Y_i(t)$ are the log returns of the individual assets in the portfolio. They are related to a common factor $X(t)$ in the following manner:

$$Y_i(t) = \alpha X(t) + \xi_i(t), \quad i = 1, ..., N$$  \hspace{1cm} (D.2)

$$X(t) = \varepsilon(t), \quad \varepsilon(t) \sim \mathcal{N}(0, \sigma^2_x, t)$$  \hspace{1cm} (D.3)

$$\log(\sigma^2_{x,t}) = a_0 + a_1 a \varepsilon_{t-1} + b_1 \log(\sigma^2_{x,t-1})$$  \hspace{1cm} (D.4)

Unless otherwise specified the parameters used are: $\mu = 0$, $a_0 = -0.7$, $a_1a = -10.3$, $a_1b = 0.15$, $b_1 = 0.92$.

Recall from section 2.3.3 that the unconditional variance $\sigma^2$ of an EGARCH(1,1) process is equal to (see [17]):

$$\sigma^2 = \exp \left( \frac{a_0 - a_1b \sqrt{2/\pi}}{1 - b_1} + \frac{1}{2} \left( \frac{a_{1a}^2 + a_{1b}^2}{2 - 1 - b_1^2} \right) \right) \prod_{m=0}^{\infty} \left[ F_m(a_{1a}, a_{1b}, b_1) + F_m(-a_{1a}, a_{1b}, b_1) \right]$$  \hspace{1cm} (D.5)

where $F_m$ is defined as:

$$F_m = N \left[ b_1^m (a_{1b} - a_{1a}) \right] \exp(b_1^{2m} a_{1a} a_{1b})$$  \hspace{1cm} (D.6)

in which $N[.]$ is the cumulative standard normal distribution.
D.2 Exponential Decay

As can be deduced from the form of equation here above, the decay of the leverage function with \( \tau \) is exponential (see figure D.1).

![Graph showing exponential decay of leverage function with \( \tau \)](image)

\[
L(\tau), \ \tau > 0
\]

\[
\text{Stocks} \quad -0.0086 \exp(-\frac{\tau}{61}) \quad -0.264 \exp(-\frac{\tau}{10})
\]

**Figure D.1:** Exponential decay of the leverage function with \( \tau \) when using the EGARCH(1,1) specification outline in equations ?? and 2.32.

D.3 Correlation between Assets

The correlation between the assets in the inverse model is derived as follows:

\[
corr(Y_i, Y_j) = \frac{E[(Y_i - E[Y_i])(Y_j - E[Y_j])]}{\sqrt{E[Y_i^2]E[Y_j^2]}} = \frac{E[Y_iY_j]}{\sqrt{E[Y_i^2]E[Y_j^2]}} = \frac{E[a^2X^2 + a\zeta_iX + a\zeta_jX + \zeta_i\zeta_j]}{\sqrt{E[a^2X^2 + 2a\zeta_iX + \zeta_i^2]E[a^2X^2 + 2a\zeta_jX + \zeta_j^2]}}
\]

\[
= \frac{E[a^2X^2] + E[\zeta_i\zeta_j]}{\sqrt{(a^2E[X^2] + E[\zeta_i^2])(a^2E[X^2] + E[\zeta_j^2])}}
\]

\[
= \frac{E[a^2X^2]}{\sqrt{(a^2E[X^2] + E[\zeta_i^2])(a^2E[X^2] + E[\zeta_j^2])}}
\]

If we set \( E[X^2] = \sigma_x^2 \) and set the EGARCH parameters such that \( E[\zeta_i^2] = \)
D.4. Correlation in Inverse Model

\[ E[\sigma_i^2] = \sigma_i^2: \]

\[
\text{corr}(Y_i, Y_j) = \frac{\alpha^2 \sigma_x^2}{\sqrt{(\alpha^2 \sigma_x^2 + \sigma_i^2)(\alpha^2 \sigma_x^2 + \sigma_j^2)}} = \frac{\alpha^2 \sigma_x^2}{\alpha^2 \sigma_x^2 + \sigma_i^2}
\]

(D.8)

Finally, if we set \( \sigma_i^2 \) and \( \sigma_x^2 \) equal one to another:

\[
\text{corr}(Y_i, Y_j) = \frac{\alpha^2}{\alpha^2 + \sigma_i^2 / \sigma_x^2} = \frac{\alpha^2}{\alpha^2 + 1}
\]

(D.9)

\[ \text{D.4 Correlation in Inverse Model} \]

The correlation between the assets in the inverse model is derived as follows:

\[
\text{corr}(Y_i, Y_j) = \frac{E[(Y_i - E[Y_i])(Y_j - E[Y_j])]}{\sqrt{E[Y_i^2]E[Y_j^2]}} = \frac{E[Y_i Y_j]}{\sqrt{E[Y_i^2]E[Y_j^2]}}
\]

\[
= \frac{E[\alpha^2 X^2 + a_1 \eta_i X + a_{11} \eta_i \eta_j]}{\sqrt{E[\alpha^2 X^2 + 2a_1 \eta_i X + \eta_i^2 ]E[\alpha^2 X^2 + 2a_1 \eta_j X + \eta_j^2 ]}}
\]

\[
= \frac{E[\alpha^2 X^2]}{\sqrt{(E[\alpha^2 X^2] + E[\eta_i^2])(E[\alpha^2 X^2] + E[\eta_j^2])}}
\]

(D.10)

If we set \( E[X^2] = \sigma_x^2 \) and set the EGARCH parameters such that \( E[\eta_i^2] = E[\eta_j^2] = \sigma_i^2 \):

\[
\text{corr}(Y_i, Y_j) = \frac{\alpha^2 \sigma_x^2}{\sqrt{(\alpha^2 \sigma_x^2 + \sigma_i^2)(\alpha^2 \sigma_x^2 + \sigma_j^2)}} = \frac{\alpha^2 \sigma_x^2}{\alpha^2 \sigma_x^2 + \sigma_i^2}
\]

(D.11)

Finally, if we set \( \sigma_i^2 \) and \( \sigma_x^2 \) equal one to another:

\[
\text{corr}(Y_i, Y_j) = \frac{\alpha^2}{\alpha^2 + \sigma_i^2 / \sigma_x^2} = \frac{\alpha^2}{\alpha^2 + 1}
\]

(D.12)
D.5 Time Decay of the Leverage Function

Changing the volatility autocorrelation parameter $b_1$ from 0.8 to 0.95 leads to a change in the time decay of the leverage function (see figure D.2).

Gain/loss asymmetry and the maximum leverage effect increase as $b_1$ is increased. The drift in the optimal investment horizons when $b_1$ is varied from 0.8 to 0.95 can be seen in the left plot of figure D.3. $\tau^+_{\rho}$ drifts from 16 to 28 days, whereas $\tau^-_{\rho}$ drops from 8 to 7 days. The maximum of the leverage function of the artificial index of 10 assets rises sharply as $b_1$ is varied from 0.8 to 0.95 (right plot of figure D.3). The maximum of the leverage effect moves from $\max L(\tau) \sim 0.365$ to $\max L(\tau) \sim 0.475$.

Figure D.2: Change in the time decay of the leverage function when the volatility autocorrelation parameter $b_1$ is varied from 0.8 to 0.95, by steps of 0.025. Note that the maximum of the leverage function also increases with $b_1$. 

![Image of Figure D.2](image-url)
Figure D.3: (top left) Asymmetry increases with $b_1$. (top right) Increasing the volatility autocorrelation parameter $b_1$ leads to an increase in the maximum leverage effect. (lower left) Optimal investment horizons for gains drift away from 16 to 28 days. Red to purple corresponds to $b_1 = 0.8 \rightarrow 0.95$. (lower right) Optimal investment horizons for losses decrease from 8 to 7 days. Red to purple corresponds to $b_1 = 0.8 \rightarrow 0.95$. 

D.5. Time Decay of the Leverage Function


